Another Proof of Hadamard’s Determinantal Inequality

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Abstract. We offer a new proof of Hadamard’s celebrated inequality for determinants of positive matrices that is based on a simple identity, which may be of independent interest.

A hermitian $n \times n$ matrix $A$ is said to be positive, if, for all $n \times 1$ vectors $x$, $x^*Ax > 0$ unless $x$ is the zero vector. Thus, if $A$ is positive, all of its principal sub-matrices are also positive. Moreover, $A$ is positive if and only if the determinants of all these sub-matrices are positive. In particular, if $A = [a_{ij}]$ is positive, then all of its diagonal entries, $a_{11}, a_{22}, \ldots, a_{nn}$, and its determinant, $\det A$, are positive. These are well-known facts about positive matrices that can be found in most textbooks on Matrix Analysis, such as, for instance, [1] and [3].

In 1893, Hadamard [2] discovered a fundamental fact about positive matrices, viz., that, for such $A = [a_{ij}]$,

$$\det A \leq a_{11}a_{22} \cdots a_{nn}.$$ 

Our purpose here is to present another proof of Hadamard’s inequality which is based on the following identity.

Lemma 1. Suppose $A$ is an $n \times n$ matrix, $\tilde{A}$ is its cofactor matrix, and $x, y$ are $n \times 1$ vectors. Then

$$\det A - \det \begin{bmatrix} A & x \\ y^t & 1 \end{bmatrix} = x^t\tilde{A}y.$$

Proof. Identify $\mathbb{C}^n$ with the space of $n \times 1$ vectors with complex entries, and consider the bilinear form

$$B(x, y) = \det A - \det \begin{bmatrix} A & x \\ y^t & 1 \end{bmatrix}, \quad x, y \in \mathbb{C}^n.$$
Denoting the usual orthonormal basis of $\mathbb{C}^n$ by $e_1, e_2, \ldots, e_n$, it’s easy to see that

$$B(e_i, e_j) = A_{i,j},$$

the $ij$th element in $\tilde{A}$. Hence, if

$$x = \sum_{i=1}^n x_i e_i, \quad y = \sum_{i=1}^n y_i e_i \in \mathbb{C}^n,$$

then, by bilinearity,

$$B(x, y) = \sum_{i,j=1}^n x_i y_j B(e_i, e_j) = \sum_{i,j=1}^n x_i y_j A_{ij}$$

$$= \sum_{i=1}^n x_i \sum_{j=1}^n A_{ij} y_j = x^T \tilde{A} y,$$

as stated. □

As an easy consequence, we have:

**Theorem 1.** Suppose $A$ is an $n \times n$ positive matrix. Then

$$\det \begin{bmatrix} A & x \\ x^* & 1 \end{bmatrix} \leq \det A \quad (x \in \mathbb{C}^n),$$

with equality if and only if $x = 0$.

**Proof.** Since $A$ is invertible, and its inverse is also positive, it follows from the lemma that

$$\det A - \det \begin{bmatrix} A & x \\ x^* & 1 \end{bmatrix} = x^* \tilde{A} x = \det A x^* A^{-1} x \geq 0,$$

and the inequality is strict unless $x$ is the zero vector. The result follows. □

**Corollary 1.** Denoting by $A_k$ the sub-matrix of $A$ of order $k \times k$ that occupies the top left-hand corner of $A = [a_{ij}]$, then

$$\det A \leq a_{nn} \det A_{n-1},$$

and the inequality is strict unless all the entries in the last column of $A$, save the last one, are zero.
Hadamard’s classical inequality is an immediate consequence of this, viz.,

**Theorem 2** (Hadamard). If $A = [a_{ij}]$ is an $n \times n$ positive matrix, then

$$\det A \leq \prod_{i=1}^{n} a_{ii},$$

with equality if and only if $A$ is a diagonal matrix.

Coupling this with the fact that the determinant of $A$ is the product of its eigenvalues, $\lambda_1, \lambda_2, \ldots, \lambda_n$, say, we can affirm that

$$\prod_{i=1}^{n} \lambda_i \leq \prod_{i=1}^{n} a_{ii},$$

with equality if and only if $A$ is a diagonal matrix. But, also, the sum of the eigenvalues of $A$ is its trace, i.e.,

$$\sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} a_{ii}.$$

In other words, denoting by $\sigma_r(x_1, x_2, \ldots, x_n)$ the $r$th symmetric function of $n$ variables, $x_1, x_2, \ldots, x_n$, we have that

$$\sigma_r(\lambda_1, \lambda_2, \ldots, \lambda_n) \leq \sigma_r(a_{11}, a_{22}, \ldots, a_{nn}),$$

if $r = 1$ or $r = n$. It’s of interest to observe that this remains true if $1 < r < n$. For completeness, we sketch a proof of this statement.

Indeed, $\sigma_r(\lambda_1, \lambda_2, \ldots, \lambda_n)$ is the coefficient $a_r$ of $t^{n-r}$ in the polynomial

$$\prod_{i=1}^{n}(t + \lambda_i) = \det(A + tI).$$

But $a_r$ is equal to the sum of the determinants of all the $r \times r$ principal sub-matrices of $A$, which are also positive. Hence, applying Hadamard’s result to each of them, we deduce that $a_r \leq \sigma_r(a_{11}, a_{22}, \ldots, a_{nn})$ as claimed.
REFERENCES


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