Fibonacci Sequences in Groups

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1. Introduction

An ordered pair (x_1, x_2) of elements of a group G determines a sequence in G by the rule

$$x_n x_{n+1} = x_{n+2}, \quad n \in \mathbb{N}. \tag{1}$$

When this sequence is periodic, its fundamental period is called the *Fibonacci length* of (x_1, x_2) in G. When G is a finite 2-generator group, the minimum of these lengths over all generating pairs defines an invariant $\lambda(G)$ of G.

After briefly listing some known results, we launch the quest for infinite groups of finite Fibonacci length by giving three modest examples and conclude with a selection of open problems.

2. Finite Groups

The cyclic case was covered by D. D. Wall in [7]. Since the classical Fibonacci series of integers modulo 5 has fundamental period equal to 20, it follows that this is the value of $\lambda(Z_5 \times Z_5)$. It is a remarkable fact [1] that the restricted Burnside group $R(2,5)$ also has length 20. Simple groups of order less than a million are considered in [3] and, more recently, the binary polyhedral groups are studied in [2], which contains a useful list of references.

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3. Infinite Groups

Many of the Fibonacci groups themselves, which are defined by means of a presentation using the relations (1), are known to be infinite. Here we seek examples that "occur in nature".

To begin with a non-example, the Fibonacci length of the infinite cyclic group Z is not defined. Indeed, it follows from a result in [6] that for any non-zero r-tuple $(x_1, \dots, x_r) \in \mathbb{Z}^r$ with $r \in \mathbb{N}$, the sequence defined by

$$x_n + x_{n+1} + \cdots + x_{n+r-1} = x_{n+r}, \quad n \in \mathbb{N}, \tag{2}$$

is non-periodic. Thus, a necessary condition for a group to have finite Fibonacci length is that its derived group be of finite index.

Example 1. For the group

$$Z_2 * Z_2 = \langle a, b \mid a^2 = b^2 = 1 \rangle,$$

we obtain the sequence

$$a,\ b,\ ab,\ bab,\ b,\ ba,\ a,\ b, \dots,$$

showing that the infinite dihedral group has Fibonacci length at most 6. Since the Fibonacci group $F(2, n)$ (see [6]) is finite for $n \leq 5$ this length is precisely 6.

Example 2. In the case of the right-angled Coxeter group

$$C = \langle a, b, c \mid a^2 = b^2 = c^2 = (bc)^2 = 1 \rangle,$$

we take $(x_1, x_2, x_3) = (a, b, c)$ and generalize (1) to the multiplicative version of the equations (2) with $r = 3$ and so define λ_3 in analogy with $\lambda = \lambda_2$ above. The resulting sequence

$$a,\ b,\ c,\ abc,\ bcabc,\ b,\ c,\ bca,\ a,\ b,\ c, \dots$$

shows that $\lambda_3(C) \leq 8$.

Note that this group is large in the sense of [4]: the subgroup generated by $x = ab$ and $y = bc$ has index 2 and is given by the presentation

$$C^+ = \langle x, y \mid y^2 = 1 \rangle,$$

and the subgroup of C^+ generated by x and yxy again has index 2 and is free on these generators.

Our final example is rather more ambitious.

Definition. A permutation $\pi \in \mathrm{Sym}(\mathbb{Z})$ is said to be *n-periodic*, $n \in \mathbb{N}$, if

$$(i+n)\pi = i\pi + n, \quad \forall i \in \mathbb{Z},$$

and then n is a *period* of π.

Remarks. 1. For such a π, we have

$$i \equiv j(\operatorname{mod} n) \Longrightarrow i\pi \equiv j\pi(\operatorname{mod} n), \tag{3}$$

so that π acts on the residue classes modulo n and thus defines a member of the symmetric group $\mathrm{Sym}(n)$. Moreover, π is determined by its values on the residues $0, 1, \ldots, n-1$:

$$i\pi = i + a_i, \quad 0 \leq i \leq n-1,$$

where $a_i \in \mathbb{Z}$, and we call $(a_0, a_1, \ldots, a_{n-1})$ the *signature* of π.

2. If $\alpha, \beta \in \mathrm{Sym}(\mathbb{Z})$ are m-, n-periodic respectively, then $lcm\{m, n\}$ is a period of $\alpha\beta$, whence the set of all n-periodic permutations forms a group $\mathrm{Sym}_n(\mathbb{Z})$. The union of the $\mathrm{Sym}_n(\mathbb{Z})$ is the group $\mathrm{Sym}_*(\mathbb{Z})$ of periodic permutations.

3. Taking $n = 1$ in (3), we see that $\mathrm{Sym}_1(\mathbb{Z})$ is just the cyclic group generated by the successor permutation σ sending i to $i+1$ for all i. Since every cycle in every power of σ is periodic, we also see that $\mathrm{Sym}_*(\mathbb{Z})$ contains the group $\mathrm{Cyc}(\langle\sigma\rangle)$ of all modular permutations of $\mathbb{Z}$ (in the sense of [5]).

4. In the light of these remarks, a little thought shows that the group $\mathrm{Sym}_n(\mathbb{Z})$ is naturally isomorphic to the wreath product Z wr $\mathrm{Sym}(n)$.

Example 3. The periodic permutations with signatures

$$\alpha = (2, -2), \quad \beta = (3, -2, -1)$$

both belong to $\mathrm{Sym}_6(\mathbb{Z})$ and generate a subgroup H isomorphic to an extension of Z^5 by the alternating group A_6. They determine the following

Fibonacci sequence (written as a column, with parentheses and commas omitted).

2	−2	2	−2	2	−2
3	−2	−1	3	−2	−1
1	−3	0	−4	5	1
−1	−1	−4	4	−2	4
0	−5	−4	0	9	0
−1	−1	5	−1	−6	4
−1	0	−10	−1	8	4
3	−2	5	−11	2	3
2	−2	−8	4	11	−7
7	−9	3	0	4	−5
5	−7	−1	−5	11	−3
0	2	0	−5	3	0
5	−7	1	−2	6	−3
5	0	1	1	−4	−3
2	−2	2	−2	2	−2
3	−2	−1	3	−2	−1

We deduce that $\lambda(H) \leq 14$.

4. Some Open Problems

Problem 1. Re the last example, surely $\lambda(H) = 14$?

Problem 2. Is this group H torsion-free?

Problem 3. Can anything be said about the rate of growth of the sequence $\lambda(A_n)$, $n \geq 4$ where A_n is the alternating group of degree n? (It begins with 16, 12, 10.)

Problem 4. Is the containment $\mathrm{Cyc}(\langle\sigma\rangle) \leq \mathrm{Sym}_*(\mathbb{Z})$ proper?

Problem 5. Does there exist a large group $G = \langle x_1, x_2 \rangle$ in which the Fibonacci length of (x_1, x_2) is finite?

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