Weyl type Theorems and the Approximate Point Spectrum

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Abstract. It is shown that, if an operator $T$ on a complex Banach space or its adjoint $T^*$ has the single-valued extension property, then the generalized a-Browder’s theorem holds for $f(T)$ for every complex-valued analytic function $f$ on a neighborhood of the spectrum of $T$. We also study the generalized a-Weyl’s theorem in connection with the single-valued extension property. Finally, we examine the stability of the generalized a-Weyl’s theorem under commutative perturbations by finite rank operators.

1. Introduction

Throughout this paper $X$ will denote an infinite-dimensional complex Banach space and $\mathcal{L}(X)$ the unital (with unit the identity operator, $I$, on $X$) Banach algebra of all bounded linear operators acting on $X$. For an operator $T \in \mathcal{L}(X)$, let $T^*$ denote its adjoint, $N(T)$ its kernel, $R(T)$ its range, $\sigma(T)$ its spectrum, $\sigma_a(T)$ its approximate point spectrum, $\sigma_{sa}(T)$ its surjective spectrum and $\sigma_p(T)$ its point spectrum. For a subset $K$ of $\mathbb{C}$ we write $\text{iso}(K)$ for its isolated points and $\text{acc}(K)$ for its accumulation points.

From [14] we recall that for $T \in \mathcal{L}(X)$, the ascent $a(T)$ and the descent $d(T)$ are given by

$$a(T) = \inf\{n \geq 0 : N(T^n) = N(T^{n+1})\}$$

and

$$d(T) = \inf\{n \geq 0 : R(T^n) = R(T^{n+1})\},$$

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respectively; the infimum over the empty set is taken to be $\infty$. If the ascent and the descent of $T \in \mathcal{L}(X)$ are both finite then $a(T) = d(T) = p$, $X = N(T^p) \oplus R(T^p)$ and $R(T^p)$ is closed.

For $T \in \mathcal{L}(X)$ we will denote by $\alpha(T)$ the nullity of $T$ and by $\beta(T)$ the defect of $T$. If the range $R(T)$ of $T$ is closed and $\alpha(T) < \infty$ (resp., $\beta(T) < \infty$) then $T$ is called an upper semi-Fredholm (resp., a lower semi-Fredholm) operator. If $T \in \mathcal{L}(X)$ is either upper or lower semi-Fredholm, then $T$ is called a semi-Fredholm operator, and the index of $T$ is defined by $\text{ind}(T) = \alpha(T) - \beta(T)$. If both $\alpha(T)$ and $\beta(T)$ are finite then $T$ is called a Fredholm operator. For a $T$-invariant closed linear subspace $Y$ of $X$, let $T \mid Y$ denote the operator given by the restriction of $T$ to $Y$.

For a bounded linear operator $T$ and for each integer $n$, define $T_n$ to be the restriction of $T$ to $R(T^n)$ viewed as a map from $R(T^n)$ into itself. If for some integer $n$ the range space $R(T^n)$ of $T$ is closed and $T_n = T \mid R(T^n)$ is an upper (resp., lower) semi-Fredholm operator then $T$ is called an upper (resp., lower) semi-B-Fredholm operator. Moreover if $T_n$ is a Fredholm operator, then $T$ is called a B-Fredholm operator. In this situation, from [1, Proposition 2.1], $T_m$ is a Fredholm operator and $\text{ind}(T_m) = \text{ind}(T_n)$ for each $m \geq n$ which permits to define the index of a B-Fredholm operator $T$ as the index of the Fredholm operator $T_n$, where $n$ is any integer such that $R(T^n)$ is closed and $T_n$ is a Fredholm operator. Let $BF(X)$ be the class of all B-Fredholm operators and $\rho_{BF}(T) = \{ \lambda \in \mathbb{C} : T - \lambda I \in BF(X) \}$ be the B-Fredholm resolvent of $T$ and let $\sigma_{BF}(T) = \mathbb{C} \setminus \rho_{BF}(T)$ the B-Fredholm spectrum of $T$. The class $BF(X)$ has been studied by M. Berkani (see [1, Theorem 2.7]), where it was shown that $T \in \mathcal{L}(X)$ is a B-Fredholm operator if and only if $T = T_0 \oplus T_1$ where $T_0$ is a Fredholm operator and $T_1$ is a nilpotent one. He also proved that $\sigma_{BF}(T)$ is a closed subset of $\mathbb{C}$ and showed that the spectral mapping theorem holds for $\sigma_{BF}(T)$, that is, $f(\sigma_{BF}(T)) = \sigma_{BF}(f(T))$ for any complex-valued analytic function on a neighborhood of the spectrum $\sigma(T)$.

An operator $T \in \mathcal{L}(X)$ is called a Weyl operator if it is Fredholm of index 0, a Browder operator if it is Fredholm of finite ascent and descent and a B-Weyl operator if it is B-Fredholm of index 0. The Weyl spectrum, the Browder spectrum and the B-Weyl spectrum of $T$ are defined by

$$\sigma_w(T) = \{ \lambda \in \mathbb{C} : T - \lambda I \text{ is not Weyl} \},$$

$$\sigma_B(T) = \{ \lambda \in \mathbb{C} : T - \lambda I \text{ is not Browder} \},$$

$$\sigma_{BF}(T) = \{ \lambda \in \mathbb{C} : T - \lambda I \text{ is not B-Fredholm} \}.$$
\[\sigma_b(T) = \{ \lambda \in \mathbb{C} : T - \lambda I \text{ is not Browder} \},\]
\[\sigma_{BW}(T) = \{ \lambda \in \mathbb{C} : T - \lambda I \text{ is not B-Weyl} \},\]
respectively. We will denote by \(E(T)\) (resp. \(E^a(T)\)) the set of all eigenvalues of \(T\) which are isolated in \(\sigma(T)\) (resp., \(\sigma_a(T)\)) and by \(E_0(T)\) (resp. \(E^a_0(T)\)) the set of all eigenvalues of \(T\) of finite multiplicity which are isolated in \(\sigma(T)\) (resp., \(\sigma_a(T)\)).

Let \(SF(X)\) be the class of all semi-Fredholm operators on \(X\), \(SF_+(X)\) the class of all upper semi-Fredholm operators on \(X\) and \(SF_-(X)\) the class of all \(T \in SF_+(X)\) such that \(\text{ind}(T) \leq 0\). For \(T \in \mathcal{L}(X)\), let
\[\sigma_{SF}(T) = \{ \lambda \in \mathbb{C} : T - \lambda I \notin SF(X) \},\]
\[\sigma_{SF_+}(T) = \{ \lambda \in \mathbb{C} : T - \lambda I \notin SF_+(X) \},\]
\[\rho_{SF}(T) = \mathbb{C} \setminus \sigma_{SF}(T) \text{ and } \rho_{SF_+}(T) = \mathbb{C} \setminus \sigma_{SF_+}(T).\]

Similarly, let \(SBF(X)\) be the class of all semi-B-Fredholm operators on \(X\), \(SBF_+(X)\) the class of all upper semi-B-Fredholm operators on \(X\) and \(SBF_-(X)\) the class of all \(T \in SBF_+(X)\) such that \(\text{ind}(T) \leq 0\). For \(T \in \mathcal{L}(X)\), the sets \(\sigma_{SBF}(T), \rho_{SBF}(T), \sigma_{SBF_+}(T)\) and \(\rho_{SBF_+}(T)\) are defined in an obvious way.

An operator \(T \in \mathcal{L}(X)\) is called semi-regular if \(R(T)\) is closed and \(N(T) \subseteq R(T^n)\) for every \(n \in \mathbb{N}\). The semi-regular resolvent set is defined by \(s\text{-reg}(T) = \{ \lambda \in \mathbb{C} : T - \lambda I \text{ is semi-regular} \}\), we note that \(s\text{-reg}(T) = s\text{-reg}(T^*)\) is an open subset of \(\mathbb{C}\). As a consequence of [8, Théorème 2.7], we obtain the following result.

**Proposition 1.1.** Let \(T \in \mathcal{L}(X)\).

(i) If \(T\) has the SVEP then \(s\text{-reg}(T) = \rho_a(T) = \mathbb{C} \setminus \sigma_a(T)\).

(ii) If \(T^*\) has the SVEP then \(s\text{-reg}(T) = \rho_{su}(T) = \mathbb{C} \setminus \sigma_{su}(T)\).

We recall that an operator \(T \in \mathcal{L}(X)\) has the single-valued extension property, abbreviated SVEP, if, for every open set \(U \subset \mathbb{C}\), the only analytic solution \(f : U \rightarrow X\) of the equation \((T - \lambda I)f(\lambda) = 0\) for all \(\lambda \in U\) is the zero function on \(U\). We will denote by \(H(\sigma(T))\) the set of all complex-valued functions which are analytic on an open set containing \(\sigma(T)\).

The remainder of the following deals with Riesz points and left poles. A complex number \(\lambda\) is said to be Riesz point of \(T \in \mathcal{L}(X)\) if \(\lambda \in \text{iso}(\sigma(T))\) and the corresponding spectral projection is of finite-dimensional range. The set of all Riesz points of \(T\) will be denoted by
It is known that if $T \in \mathcal{L}(X)$ and $\lambda \in \sigma(T)$, then $\lambda \in \Pi_0(T)$ if and only if $T - \lambda I$ is Fredholm of finite ascent and descent (see [3]). Consequently $\sigma_0(T) = \sigma(T) \setminus \Pi_0(T)$. We will denote by $\Pi(T)$ the set of all poles of the resolvent of $T$. A complex number $\lambda \in \sigma_a(T)$ is said to be a left pole of $T$ if $a(T - \lambda I) < \infty$ and $R((T - \lambda I)^{a(T - \lambda I) + 1})$ is closed, and that it is a left pole of $T$ of finite rank if it is a left pole of $T$ and $a(T - \lambda I) < \infty$. We will denote by $\Pi^a(T)$ the set of all left poles of $T$, and by $\Pi^0(T)$ the set of all left poles of $T$ of finite rank. If $\lambda \in \Pi^a(T)$, then it is easily seen that $T - \lambda I$ is an operator of topological uniform descent, therefore from [4], it follows that $\lambda$ is isolated in $\sigma_a(T)$ [2, Theorem 2.5]. Let $T \in \mathcal{L}(X)$ and $\lambda \in \mathbb{C}$ be isolated in $\sigma_a(T)$; then $\lambda \in \Pi^a(T)$ if and only if $\lambda \notin \sigma_{SBF}^-(T)$, and $\lambda \in \Pi^0_0(T)$ if and only if $\lambda \notin \sigma_{SF}^-(T)$.

For $T \in \mathcal{L}(X)$ we will say that:

(i) $T$ satisfies Weyl’s theorem if $\sigma_w(T) = \sigma(T) \setminus E_0(T)$;
(ii) $T$ satisfies generalized Weyl’s theorem if $\sigma_{BW}(T) = \sigma(T) \setminus E(T)$;
(iii) $T$ satisfies a-Weyl’s theorem if $\sigma_{SF}^-(T) = \sigma_a(T) \setminus E_0^a(T)$;
(iv) $T$ satisfies generalized a-Weyl’s theorem if $\sigma_{SBF}^-(T) = \sigma_a(T) \setminus E^a(T)$;
(v) $T$ satisfies Browder’s theorem if $\sigma_{w}(T) = \sigma(T) \setminus \Pi_0(T)$;
(vi) $T$ satisfies generalized Browder’s theorem if $\sigma_{BW}(T) = \sigma(T) \setminus \Pi(T)$;
(vii) $T$ satisfies a-Browder’s theorem if $\sigma_{SF}^-(T) = \sigma_a(T) \setminus \Pi_0^a(T)$;
(viii) $T$ satisfies generalized a-Browder’s theorem if $\sigma_{SBF}^-(T) = \sigma_a(T) \setminus \Pi^a(T)$.

Before proving our main result we deal with some preliminary results.
Proposition 1.2. Let $T \in \mathcal{L}(X)$.

(i) If $T$ has the SVEP then $\text{ind}(T - \lambda I) \leq 0$ for every $\lambda \in \rho_{\text{SBF}}(T)$.

(ii) If $T^*$ has the SVEP then $\text{ind}(T - \lambda I) \geq 0$ for every $\lambda \in \rho_{\text{SBF}}(T)$.

Proof. (i) Let $\lambda \in \rho_{\text{SBF}}(T)$, then there exists an integer $p$ such that $(T | R(T - \lambda I)^p) - \lambda I = (T - \lambda I) | R(T - \lambda I)^p$ is semi-Fredholm. From the Kato decomposition, there exists $\delta > 0$ such that

$$\{ \lambda \in \mathbb{C} : 0 < |\mu - \lambda| < \delta \} \subseteq s\text{-reg}(T | R(T - \lambda I)^p).$$

Since $T$ has the SVEP, Proposition 1.1 implies that $s\text{-reg}(T | R(T - \lambda I)^p) = \rho_a(T | R(T - \lambda I)^p)$. Therefore, $N((T | R(T - \lambda I)^p) - \rho_a(T | R(T - \lambda I)^p) = 0$ and so

$$\text{ind}(T - \lambda I) = \text{ind}((T | R(T - \lambda I)^p) - \rho_a(T | R(T - \lambda I)^p) \leq 0,$$

holding for $0 < |\mu - \lambda| < \delta$. Thus, by the continuity of the index we obtain $\text{ind}(T - \lambda) \leq 0$.

(ii) Follows by similar reasoning, and may also be derived from the first assertion and the fact that $\text{ind}(T^*) = -\text{ind}(T)$. □

Corollary 1.3. Let $T$ be a bounded linear operator on $X$. If $T^*$ has the SVEP, then $\sigma_{\text{SF}}(T) = \sigma_w(T)$.

Proof. We have only to show that $\sigma_w(T) \subseteq \sigma_{\text{SF}}(T)$, since the other inclusion is always verified. Let $\lambda$ be given in $\rho_{\text{SF}}(T)$, then $T - \lambda I$ is semi-Fredholm and $\text{ind}(T - \lambda I) \leq 0$. Since $T^*$ has the SVEP, Proposition 1.2 implies that $\text{ind}(T - \lambda I) \geq 0$, and hence $\text{ind}(T - \lambda I) = 0$, which proves that $T - \lambda I$ is Fredholm of index 0 and $\lambda \in \rho_w(T)$. □

The following results relate the generalized a-Weyl’s theorem and the generalized a-Browder’s theorem to the single-valued extension property. As motivation for the proofs, we use some ideas in [10, 12].

Proposition 1.4. Let $T$ be a bounded linear operator on $X$.

(i) If $T^*$ has the SVEP, then $T$ satisfies generalized a-Weyl’s theorem if and only if it satisfies generalized Weyl’s theorem.

(ii) If $T$ has the SVEP, then $T^*$ satisfies generalized a-Weyl’s theorem if and only if it satisfies generalized Weyl’s theorem.

Proof. (i) Since $T^*$ has the SVEP, [6, Proposition 1.3.2] implies that $\sigma(T) = \sigma_a(T)$ and consequently $E^a(T) = E(T)$. Suppose that $T$ satisfies generalized Weyl’s theorem, then $\sigma_{BW}(T) = \sigma(T) \setminus E(T)$ =
σ_a(T) \setminus E^a(T)$. Let \( \lambda \notin \sigma_{SBF^-}(T) \) be given, then \( T - \lambda I \) is semi-B-Fredholm and \( \text{ind}(T - \lambda I) \leq 0 \). Therefore, by Proposition 1.2, if follows that \( \text{ind}(T - \lambda I) = 0 \) and consequently \( T - \lambda I \) is B-Fredholm of index 0. Hence \( \lambda \notin \sigma_{BW}(T) \) and \( \sigma_{BW}(T) \subset \sigma_{SBF^-}(T) \). Since the opposite inclusion is clear, we conclude that indeed \( \sigma_{SBF^-}(T) = \sigma_{BW}(T) = \sigma_a(T) \setminus E^a(T) \) which proves the equivalence between generalized Weyl’s theorem and generalized a-Weyl’s theorem for \( T \).

(ii) Similar to the proof of the first assertion. \(\square\)

Our main result reads now as follows.

\textbf{Theorem 1.5.} Let \( T \) be a bounded linear operator on \( X \). If \( T \) or its adjoint \( T^* \) satisfies the SVEP, then generalized a-Browder’s theorem holds for \( f(T) \) for every \( f \in H(\sigma(T)) \).

\textbf{Proof.} Let us establish that generalized a-Browder’s theorem holds for \( T \). If \( T^* \) has the SVEP, then by [12, Theorem 2.4], it follows that a-Browder’s theorem holds for \( T \), and consequently Browder’s theorem holds for \( T \). Thus \( \sigma_{SF^+}(T) = \sigma_a(T) \setminus \Pi_0^a(T) \) and \( \sigma_b(T) = \sigma(T) \setminus \Pi_0(T) \). Moreover, since \( \sigma_a(T) = \sigma(T) \), \( \Pi_0^a(T) = \Pi_0(T) \), it follows that \( \sigma_{SF^+}(T) = \sigma(T) \setminus \Pi_0(T) \). Because \( \sigma_{SF^+}(T) = \sigma_a(T) \), see Corollary 1.3, it follows that \( \sigma_{SF^+}(T) = \sigma(T) \setminus \Pi_0(T) = \sigma_a(T) = \sigma_b(T) \).

Let \( \lambda \in \Pi^a(T) \) be given; then \( \lambda \) is isolated in \( \sigma_a(T) \) and by [2, Theorem 2.8], it follows that \( \lambda \notin \sigma_{SBF^-}(T) \) which shows that \( \Pi^a(T) \subseteq \sigma_a(T) \setminus \sigma_{SBF^-}(T) \). Conversely if \( \lambda \in \sigma_a(T) \setminus \sigma_{SBF^-}(T) \), then \( T - \lambda I \) is semi-B-Fredholm and \( \text{ind}(T - \lambda I) \leq 0 \). Then, since \( T^* \) has the SVEP, Proposition 1.2 gives \( \text{ind}(T - \lambda I) = 0 \). Therefore \( T - \lambda I \) is Fredholm and \( \lambda \notin \sigma_a(T) = \sigma_b(T) \) which shows that \( \lambda \notin \Pi_0(T) \).

Consequently \( \lambda \) is isolated in \( \sigma_a(T) \) and hence \( \lambda \in \Pi^a(T) \). Thus \( \sigma_a(T) \setminus \sigma_{SBF^-}(T) \subset \Pi^a(T) \) and generalized a-Browder’s theorem holds for \( T \). Now if \( T \) has the SVEP, let \( \lambda \in \sigma_a(T) \setminus \sigma_{SBF^-}(T) \); \( \lambda \in \rho_{SBF^-}(T) \), then there exists an integer \( p \) such that \( R(T - \lambda I)^p \) is closed and \( (T \mid R(T - \lambda I)^p) - \lambda I = (T - \lambda I) \mid R(T - \lambda I)^p \) is a semi-Fredholm operator. Then, by the Kato decomposition, there exists \( \delta > 0 \) for which

\[
\{ \mu \in \mathbb{C} : 0 < |\mu - \lambda| < \delta \} \subseteq s\text{-reg}(T \mid R(T - \lambda I)^p) \cap \rho_{SF}(T \mid R(T - \lambda I)^p).}
\]
Since $T$ has the SVEP, so does $T | R(T - \lambda I)^p$. Therefore
\[ \text{s-reg}(T | R(T - \lambda I)^p) = \rho_a(T | R(T - \lambda I)^p) \]
and
\[ \{ \mu \in \mathbb{C} : 0 < |\mu - \lambda| < \delta \} \subseteq \rho_a(T | R(T - \lambda I)^p) \cap \rho_{SF}(T | R(T - \lambda I)^p), \]
hence $\lambda \in \text{iso}(\sigma_a(T) \cap \rho_{SF}(T))$. By [2, Theorem 2.8], it follows that $\lambda \in \Pi_a(T)$ and $\sigma_a(T) \setminus \sigma_{SF}^{-}(T) \subset \Pi_a(T)$. Since the other inclusion is clear we get $\sigma_a(T) \setminus \sigma_{SF}^{+}(T) = \Pi_a(T)$ and thus generalized a-Browder’s theorem holds for $T$. Finally, if $f \in H(\sigma(T))$, by [6, Theorem 3.3.6] $f(T)$ or $f(T^*)$ satisfies the SVEP and the above argument implies that generalized a-Browder’s theorem holds for $f(T)$. □

From Theorem 1.5 we obtain the following useful consequence.

**Corollary 1.6.** Let $T$ be a bounded linear operator on $X$. If $T$ or $T^*$ has the SVEP then generalized a-Weyl’s theorem holds for $T$ if and only if $E^a(T) = \Pi^a(T)$.

**Proof.** We only have to use the fact that an operator $T$ satisfying generalized a-Browder’s theorem, satisfies generalized a-Weyl’s theorem if and only if $\Pi^a(T) = E^a(T)$. □

In [7] the class of the operators $T \in \mathcal{L}(X)$ for which $K(T) = \{0\}$ was studied and it was shown that for such operators, the spectrum is connected and the single-valued extension property is satisfied.

**Proposition 1.7.** Let $T \in \mathcal{L}(X)$. If there exists a complex number $\lambda$ for which $K(T - \lambda I) = \{0\}$ then $f(T)$ satisfies generalized a-Browder’s theorem for every $f \in \mathcal{H}(\sigma(T))$. Moreover, if in addition, $N(T - \lambda I) = \{0\}$, then generalized a-Weyl’s theorem holds for $f(T)$ for any $f \in \mathcal{H}(\sigma(T))$.

**Proof.** Let $f$ be a non-constant complex-valued analytic function on an open neighborhood of $\sigma(T)$. Since $T$ has the SVEP so does $f(T)$ and by Theorem 1.5 generalized a-Browder’s theorem holds for $f(T)$. Now assume that $N(T - \lambda I) = \{0\}$ and $\beta \in \sigma(f(T))$ then $f(z) - \beta I = P(z)g(z)$ where $g$ is complex-valued analytic function on a neighborhood of $\sigma(T)$ without any zeros in $\sigma(T)$ while $P$ is a
complex polynomial of the form \( P(z) = \prod_{i=1}^{n} (z - \lambda_i)^{p_i} \) with distinct roots \( \lambda_1, \ldots, \lambda_n \in \sigma(T) \). Since \( g(T) \) is invertible, we have

\[
N(f(T) - \beta I) = N(P(T)) = \bigoplus_{i=1}^{n} N(T - \lambda_i I)^{p_i}.
\]

On the other hand, \([7, \text{Proposition 2.1}]\) ensures that \( \sigma_p(T) \subseteq \{ \lambda \} \) and since \( T - \lambda I \) is injective, we deduce that \( \sigma_a(T) = \emptyset \). Consequently \( N(f(T) - \beta I) = \{ \emptyset \} \) which proves that \( \sigma_p(f(T)) = \emptyset \). Thus \( E^a(f(T)) = \Pi^a(f(T)) = \emptyset \) and generalized a-Weyl’s theorem holds for \( f(T) \).

**Proposition 1.8.** Let \( T \) be a bounded linear operator on \( X \) satisfying the SVEP. If \( T - \lambda I \) has finite descent at every \( \lambda \in E^a(T) \), then \( T \) obeys generalized a-Weyl’s theorem.

**Proof.** Let \( \lambda \in E^a(T) \), then \( p = d(T - \lambda I) < \infty \) and since \( T \) has the SVEP it follows (see \([13, \text{Proposition 3}]\)) that \( a(T - \lambda I) = d(T - \lambda I) = p \) and by \([5, \text{Satz 101.2}]\), \( \lambda \) is a pole of the resolvent of \( T \) or order \( p \), consequently \( \lambda \) is an isolated point in \( \sigma_a(T) \). Then \( X = K(T - \lambda I) \oplus H_0(T - \lambda I) \), with \( K(T - \lambda I) = R(T - \lambda I)^p \) is closed, therefore \( \lambda \in \Pi^a(T) \).

Now let us consider the class \( \mathcal{P}(X) \) defined as those operators \( T \in \mathcal{L}(X) \) for which for every complex number \( \lambda \) there exists a positive integer \( p_{\lambda} \) such that \( H_0(T - \lambda I) = N(T - \lambda I)^{p_{\lambda}} \). This class has been introduced and studied in \([10, 11]\), it was shown that it contains every M-hyponormal, log-hyponormal, p-hyponormal and totally paranormal operator. It was also established that the SVEP is shared by all the operators lying in \( \mathcal{P}(X) \) and generalized Weyl’s theorem holds for \( f(T) \) whenever \( T \in \mathcal{P}(X) \) and \( f \in \mathcal{H}(\sigma(T)) \).

**Proposition 1.9.** Let \( T \in \mathcal{P}(X) \) be such that \( \sigma(T) = \sigma_a(T) \) then generalized a-Weyl’s theorem holds for \( f(T) \) for every \( f \in \mathcal{H}(\sigma(T)) \).

**Proof.** By the spectral mapping theorem for the spectrum and the approximate point spectrum, and the fact that \( f(T) \in \mathcal{P}(X) \), it suffices to establish generalized a-Weyl’s theorem for \( T \). Since \( \sigma(T) = \sigma_a(T) \) it follows that

\[
E^a(T) = \sigma_p(T) \cap \text{iso}(\sigma_a(T)) = \sigma_p(T) \cap \text{iso}(\sigma(T)) = E(T).
\]

Let \( \lambda \in E^a(T) = E(T) \), then \( X = H_0(T - \lambda I) \oplus K(T - \lambda I) \) and \( K(T - \lambda I) \) is closed. Since \( T \in \mathcal{P}(X) \), let \( p_{\lambda} \) be a positive integer.
for which \( H_0(T - \lambda I) = N(T - \lambda I)^{p_\lambda} \), therefore
\[
R(T - \lambda I)^{p_\lambda} = (T - \lambda I)^{p_\lambda} (H_0(T - \lambda I) \oplus K(T - \lambda I))
\]
\[
= (T - \lambda I)^{p_\lambda} (K(T - \lambda I))
\]
\[
= K(T - \lambda I),
\]
thus \( R(T - \lambda I)^{p_\lambda} = R(T - \lambda I)^{p_\lambda + 1} \) which by Proposition 1.8 shows that the operator \( T \) obeys generalized a-Weyl’s theorem. □

2. Generalized a-Weyl’s Theorem and Perturbation

In general, we cannot expect that generalized a-Browder’s theorem necessarily holds under finite rank perturbations. However, it does hold under commutative ones, as the following result shows.

**Theorem 2.1.** [2, Theorem 3.2] If \( T \in \mathcal{L}(X) \) is an operator satisfying generalized a-Browder’s theorem and \( F \) is a finite rank operator such that \( TF = FT \) then \( T + F \) satisfies generalized a-Browder’s theorem.

**Lemma 2.2.** Let \( T \in \mathcal{L}(X) \) be an injective operator. If \( F \) is a finite rank operator on \( X \) such that \( FT = TF \), then \( R(F) \subseteq R(T) \).

*Proof.* Since \( F \) is a finite rank operator on \( X \) there exist two systems: a system of linearly independent vectors \( e_i \) for \( i = 1, \ldots, n \) and a system of non-zero bounded linear functionals \( f_i \) for \( i = 1, \ldots, n \) on \( X \) such that
\[
F(x) = \sum_{i=1}^{n} f_i(x)e_i \quad (x \in X).
\]
Moreover, we have
\[
\sum_{i=1}^{n} f_i(x)Te_i = TF(x) = FT(x) = \sum_{i=1}^{n} f_i(Tx)e_i \quad (x \in X).
\]
On the other hand, since \( T \) is injective, it is clear that the vectors \( Te_i \) (\( 1 \leq i \leq n \)) are linearly independent. Hence \( F(x) \in \text{Vect}(\{e_1, \ldots, e_n\}) = \text{Vect}(\{Te_1, \ldots, Te_n\}) \) for all \( x \in X \). Thus \( R(F) \subseteq R(T) \), as desired. □

**Lemma 2.3.** Let \( T \in \mathcal{L}(X) \). If \( F \) is a finite rank operator on \( X \) such that \( FT = TF \) then \( \lambda \in \text{acc}(\sigma_a(T)) \) if and only if \( \lambda \in \text{acc}(\sigma_a(T+F)) \).
Proof. Let \( \lambda \notin \text{acc}(\sigma_a(T)) \) be given, there exists \( \delta > 0 \) such that if \( 0 < |\mu - \lambda| < \delta \) then \( \alpha(T - \mu I) = 0 \) and \( R(T - \mu I) \) is closed. This gives us a bounded linear operator \( S : R(T - \mu I) \to X \) such that \( S(T - \mu I) = I \) and \( (T - \mu I)S = I \mid R(T - \mu I) \). To see that \( \lambda \notin \text{acc}(\sigma_a(T + F)) \), suppose that \( \mu \in \sigma_a(T + F) \), and choose unit vectors \( x_n \in X \) such that \( (T + F - \mu I)x_n \to 0 \) as \( n \to \infty \). Let \( (x_{n(k)})_k \) be a subsequence such that \( Fx_{n(k)} \to x \in R(F) \) as \( k \to \infty \), and since this level of generality is not needed here, we may assume that \( Fx_n \to x \) as \( n \to \infty \). Therefore \( S(T + F - \mu I)x_n = x_n + SFx_n \to 0 \) as \( n \to \infty \), and since \( \lim SFx_n = Sx \) exists, it follows that \( \lim x_n = -Sx \) and consequently \( x \neq 0 \). Next observe that \( x = \lim Fx_n = -FSx \in R(F) \), then since Lemma 2.2 asserts that \( R(F) \subseteq R(T) \), we obtain \( (T - \mu I)x = -(T - \mu I)FSx = -F(T - \mu I)x = -Fx \), hence \( (T + F - \mu I)x = 0 \). Thus \( \mu \in \sigma_p(T + F) \). Finally, because eigenvectors corresponding to distinct eigenvalues of an operator are linearly independent, and since all the eigenvectors of \( T + F \) belong to the finite dimensional subspace \( R(F) \), it follows that \( \sigma_a(T + F) \) may contain only finitely many points \( \mu \) such that \( 0 < |\mu - \lambda| < \delta \), and consequently \( \lambda \notin \text{acc}(\sigma_a(T + F)) \). The opposite inclusion is similarly obtained. \( \square \)

An operator \( T \in \mathcal{L}(X) \) is said to be approximate-isoloid if any isolated point of \( \sigma_a(T) \) is an eigenvalue of \( T \).

Theorem 2.4. Let \( T \) be an approximate-isoloid operator on \( X \) that satisfies generalized a-Weyl’s theorem. If \( F \) is an operator of finite rank on \( X \) such that \( FT = TF \) then \( T + F \) satisfies generalized a-Weyl’s theorem.

Proof. Since by Theorem 2.1 generalized a-Browder’s theorem holds for \( T + F \) it suffices, from Corollary 1.5, to prove that \( E^a(T + F) = \Pi^a(T + F) \). Let \( \lambda \in E^a(T + F) \) be given, then \( \lambda \in \text{iso}(\sigma_a(T + F)) \) and \( \lambda \in \sigma_p(T + F) \), hence \( \lambda \notin \text{acc}(\sigma_a(T + F)) \) and by Lemma 2.3 \( \lambda \notin \text{acc}(\sigma_a(T)) \). We distinguish two cases. Firstly if \( \lambda \notin \sigma_a(T) \), then \( T - \lambda I \) is injective with a closed range and \( T - \lambda I \) is an upper semi-Fredholm operator on \( X \) such that \( \text{ind}(T - \lambda I) \leq 0 \), and since \( F \) is a finite rank operator on \( X \), it follows that \( T + F - \lambda I \) is an upper semi-Fredholm operator and \( \text{ind}(T + F - \lambda I) = \text{ind}(T - \lambda I) \leq 0 \). Then \( \lambda \notin \sigma_{SF^+}(T + F) \) and \( \lambda \in \Pi^a(T + F) \). On the other hand if \( \lambda \in \sigma_a(T) \), then \( \lambda \in \text{iso}(\sigma_a(T)) \) and since \( T \) is approximate-isoloid \( \lambda \in \sigma_p(T) \). Thus \( \lambda \in \text{iso}(\sigma_a(T)) \cap \sigma_p(T) = E^a(T) \). From the
fact that $T$ obeys generalized a-Weyl’s theorem, it follows that $\lambda \notin \sigma_{SBF^{-}}(T) = \sigma_{SBF^{-}}(T + F)$ and since $\lambda \in \text{iso}(\sigma_a(T + F))$, it follows that $\lambda \in \Pi_a(T + F)$. Finally $E^a(T + F) \subset \Pi_a(T + F)$, and since the reverse inclusion is verified, $T + F$ obeys generalized a-Weyl’s theorem. □

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