When Uniformly-continuous Implies Bounded

ANTHONY G. O'FARRELL[1]

1. Introduction

Let (X,ρ) and (Y,σ) be metric spaces. A function $f : X \to Y$ is (by definition) *bounded* if the image of f has finite σ-diameter. It is well-known that if X is compact then each continuous $f : X \to Y$ is bounded. Special circumstances may conspire to force all continuous $f : X \to Y$ to be bounded, without Y being compact. For instance, if Y is bounded, then that is enough. It is also enough that X be connected and that each connected component of Y be bounded. But if we ask that all continuous functions $f : X \to Y$, for arbitrary Y, be bounded, then this requires that X be compact.

What about uniformly-continuous maps? Which X have the property that each uniformly-continuous map from X into any other metric space must be bounded?

We begin with an observation.

Lemma 1.1. *Let (X,ρ) be a metric space. Then the following are equivalent:*
(1) *Each uniformly-continuous map from X into another metric space is bounded.*
(2) *Each uniformly-continuous map from X into $\mathbb{R}$ (with the usual metric) is bounded.*

Proof. Obviously (1) implies (2). The other direction follows from the facts that: (a) $f : X \to Y$ is bounded if and only if each (or any one) of the compositions

$$\sigma(b,\bullet) \circ f : x \mapsto \sigma(b, f(x)) \qquad (b \in Y)$$

is bounded, and (b) the composition $\sigma(b,\bullet)$ is uniformly-continuous if f is uniformly-continuous. $\square$

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This allows us to concentrate on the case $Y = \mathbb{R}$, with the usual metric.

1.1. **Example.** Each uniformly-continuous function $f : (a, b) \to \mathbb{R}$, mapping a bounded open interval to $\mathbb{R}$, is bounded. Indeed, given such an f, choose $\delta > 0$ with the property that the modulus of continuity $\omega_f(\delta) < 1$, i.e.,

$$|x - y| < \delta \implies |f(x) - f(y)| < 1.$$

Take $n \in \mathbb{N}$ greater than $(b - a)/\delta$, $h = (b - a)/n$, and $a_i = a + ih$ $(0 \leq i \leq n)$. Then

$$|f(x)| \leq 1 + \max\{|f(a_i)| : 1 \leq i \leq n - 1\}.$$

A very similar argument shows that if X is totally-bounded, then each uniformly-continuous function from X is bounded. However, this is not the whole story.

1.2. **Example.** Let X be the unit ball of ℓ^∞, i.e., the space of all bounded sequences $\{a_n\}$ of complex numbers, with the metric induced by the supremum norm:

$$\rho(\{a_n\}, \{b_n\}) = \sup_n |a_n - b_n|.$$

Suppose $f : X \to \mathbb{R}$ is uniformly-continuous, and choose $\delta > 0$ such that $\omega_f(\delta) < 1$. Let $m \in \mathbb{N}$ be the ceiling of $1/\delta$. Then for each $a = \{a_n\} \in X$, taking $h = \sup_n |a_n|/m$ and $b_i = iha$, we have

$$|f(a)| \leq |f(0)| + \sum_{i=1}^{m} |f(b_i) - f(b_{i-1})| \leq |f(0)| + m.$$

Thus f is bounded. However, X is not totally-bounded.

2. Epsilon-step Territories

For $\varepsilon > 0$ and $a, b \in X$, we say that a is *ε-step-equivalent to b* if there exist points $a_0 = a$, $a_1, \ldots, a_n = b$, belonging to X, with $\rho(a_{i-1}, a_i) \leq \varepsilon$ for each i. This defines an equivalence relation on X (for each fixed $\varepsilon > 0$). We call the equivalence classes *ε-step territories*, and denote the territory of a point a by $T_\varepsilon(a)$, or just $T(a)$, if the value of ε is clear from the context.

For $\varepsilon > 0$ and $a, b \in X$, we denote by $s_\varepsilon(a, b)$ (or just $s(a, b)$) the infimum of those $n \in \mathbb{N}$ (if any) for which there exist $a_0 = a$, $a_1, \ldots, a_n = b$ belonging to X, with $\rho(a_{i-1}, a_i) \leq \varepsilon$. Obviously, $s(a, b) < +\infty$ if and only if $T(a) = T(b)$.

We say that a territory $T(a)$ is *ε-step-bounded* if

$$\sup_{x \in T(a)} s(a,x) < +\infty,$$

and we call this supremum the *ε-step extent* of $T(a)$.

We define a new 'distance' function on $X \times X$, the *ε-step distance*, by setting $d_\varepsilon(a,b)$ equal to

$$\inf\Big\{\sum_{i=1}^{n} \rho(a_{i-1}, a_i)\Big\},$$

where the infimum is taken over all $a_0, a_1, \ldots, a_n \in X, a_0 = a, a_n = b$ such that $\rho(a_{i-1}, a_i) \leq \varepsilon$, and $a, b \in X$. This has all the properties of a metric, except that its value may be $+\infty$. (One may obtain a proper metric by forming $\arctan \circ d_\varepsilon$.) The distance d_ε is a proper metric when restricted to any particular ε-step territory $T(a)$. In general, $d_\varepsilon(a,b)$ is at least as large as the original $\rho(a,b)$, but $d_\varepsilon(a,b)$ coincides with $\rho(a,b)$ whenever $\rho(a,b) \leq \varepsilon$, and hence d_ε induces the same topology as ρ on T, and moreover a function $f : X \to \mathbb{R}$ is ρ-uniformly-continuous if and only if it is d_ε-uniformly-continuous. Indeed its ρ-modulus of continuity coincides with its d_ε-modulus of continuity when the argument is less than or equal to ε.

One readily checks that a territory T is ε-step-bounded if and only if its d_ε-diameter is finite. Moreover, its ε-step extent lies between

$$\frac{d_\varepsilon - \operatorname{diam}(T)}{\varepsilon} \qquad \text{and} \qquad 2 + \frac{2d_\varepsilon - \operatorname{diam}(T)}{\varepsilon}.$$

We now state the main result.

Theorem 2.1. *Let (X, ρ) be a metric space. Then the following are equivalent:*
(1) *Each uniformly-continuous function $f : X \to \mathbb{R}$ is bounded.*
(2) *For each $\varepsilon > 0$, X has only a finite number of ε-step territories, and each territory is ε-step-bounded.*

Proof. (1) $\Rightarrow$ (2): Suppose (1). Fix $\varepsilon > 0$.

Suppose that X has infinitely-many ε-step-territories. Let T_n (for $n = 1,2,3,\ldots$) be distinct territories. Then the function f, defined by

$$f(x) = \begin{cases} n & , \quad x \in T_n, \\ 0 & , \quad x \in X \sim \bigcup_{n=1}^{\infty} T_n, \end{cases}$$

is uniformly-continuous and unbounded, which is impossible. Thus X has only a finite number of ε-step territories.

Now suppose that one of the ε-step territories, say $T(a)$, is not ε-step-bounded. Define

$$g(x) = \begin{cases} d_\varepsilon(a,x) & , \quad x \in T(a), \\ 0 & , \quad x \in X \sim T(a). \end{cases}$$

Then g is uniformly-continuous on X, and unbounded, contradicting the assumption. Thus each ε-step territory is ε-step-bounded, and (2) holds.

(2) $\Rightarrow$ (1): Suppose (2), and fix $f : X \to \mathbb{R}$, uniformly-continuous.

Pick $\delta > 0$ such that $\omega_f(\delta) < 1$. With $\varepsilon = \delta$, choose $a_1,\ldots,a_n \in X$ such that $X = \bigcup_{j=1}^n T(a_j)$. Then take N to be the maximum of the ε-step extents of the $T(a_j)$, for $1 \le j \le n$. Let $M = \max_j |f(a_j)|$.

For each $x \in X$, there exists j with $x \in T(a_j)$, and then there are $x_0 = a_j$, $x_1,\ldots,x_m = x$ belonging to X, with $m \le N$ and $\rho(x_{i-1}, x_i) \le \varepsilon$. Thus

$$|f(x)| \le m + |f(a_j)| \le N + M.$$

Thus f is bounded. This proves (1). $\square$

Anthony G. O'Farrell,
Mathematics Department,
NUI, Maynooth,
Co. Kildare,
Ireland
aof@maths.may.ie

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