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Consequences of the Axiom of Blackwell Determinacy

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ABSTRACT. We describe the mathematization of games and set theoretic questions stemming from game theoretic axioms. In particular, we shall look at game theoretic axioms suggested by results and questions from statistics.

1. INTRODUCTION

1.1. **Two Versions of Game Theory.** For most mathematicians, the term "game theory" evokes associations of applications in Economics and Computer Science: it is often associated with the Prisoner's Dilemma or other applications in the social sciences. Game theory is not perceived as an area of mathematics but rather as an area in which mathematics is applicable.

But beyond the well-known applications in the social sciences, game theory has applications in a plethora of mathematical and semi-mathematical fields: games have been used successfully in several parts of mathematics, and there is a huge community of game theoretic researchers in areas of computer science.

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Even among these mathematically oriented game theorists few know much about a part of game theory that belongs to mathematical logic or, more precisely, to set theory. We are talking about the study of two-player perfect information zero-sum games of infinite length with methods from set theory that is sometimes called "Cabal-style" or "Californian" Descriptive Set Theory.

It was the Polish school of topologists and measure theorists that connected game theory to set theory. According to Steinhaus [39, 464], Banach and Mazur knew in the 1930s that there is a non-determined infinite game (constructed by a use of the Axiom of Choice) and that there is a connection between games and the Baire property.

Gale and Stewart in their seminal [7] presented the general theory of infinite games, and Mycielski and Steinhaus proposed a set theoretic analysis of game-related axioms in [32].

In the 1960s, the early set theoretic investigation of the game theoretic axioms proposed by Mycielski and Steinhaus was mainly done by Mycielski and a growing group of Californian logicians, among them John Addison, Tony Martin, Yiannis Moschovakis and Bob Solovay at the University of California at Berkeley and Los Angeles. In the 1970s, the Los Angeles area set theory seminar with researchers from UCLA and CalTech (including prominently the Californian researchers mentioned earlier plus Alekos Kechris, John Steel, and later Hugh Woodin) became known as "the Cabal" and its regular meetings together with a conference series called the *Very Informal Gathering* produced a theory that is now known as *Cabal-style Descriptive Set Theory* and was published in the four proceedings volumes of the Cabal seminar [18, 15, 16, 17] and codified in Moschovakis' text book [29].

The Cabal has investigated the consequences of game theoretic axioms in set theory, and they have unveiled very deep connections between these axioms and the foundations of mathematics and metamathematics. On their way, they have also developed a lot of set theoretic techniques for dealing with infinite games.

The entire Cabal-style theory has been developed in the setting of two-player perfect information games. Although the Cabal has been playing around with the definitions and has been looking at variants of determinacy with different sets of possible moves ($\mathsf{AD}_{\mathbb{R}}$) and variants of determinacy for games of different transfinite lengths

[38, 34], they didn't give up the general perfect information structure.[1] A reason for this is that a lot of the techniques used by the Cabal are based in the fact that perfect information strategies come from trees (in the sense that will be made clear in an example in Section 1.2), and Cabal-style set theory tends to view tree representations as a necessary condition for a set to be tractable.[2]

The game theoretic axiom that has played the central rôle in this development is the *Axiom of Determinacy* AD introduced by Mycielski and Steinhaus in their [32]. We shall define this axiom in Section 2 and give an overview of its consequences in Section 3.

There is little if any interaction between the communities of game theorists outside of set theory and the Cabal-style set theorists. Cabal-style set theorists rarely know more than the average mathematician about applied game theory and only few applied game theorists know that Cabal-style set theory even exists.

In this paper, we shall extend results from Cabal-style set theory to a broader class of games, namely to a type of stochastic games that is used in statistics and that we shall call *Blackwell games.* We shall describe how results of Cabal-style set theory can be reproduced with game theoretic axioms from statistics.

One of the objectives of this paper is to make the existence of a beautiful set theoretic structure theory connected to game theoretic axioms known to a broader public, and thereby to incite the development of more connections between Cabal-style set theory and the brands of game theory in several areas similar to the results mentioned in this paper.

1.2. **Mathematizing Games.** Historically, set theory and the mathematization of games has been connected from the very beginning. In a first mathematical analysis of the game of chess which was published in a paper entitled *Über eine Anwendung der Mengenlehre auf die Theorie des Schachspiels* [45], Zermelo essentially argued that one of the two players has an (at least) drawing strategy.

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[1] *Cf.* Steel's discussion of possible variations of determinacy axioms in [38, p. 96–97].

[2] "[Tree representations] are important in Descriptive Set Theory because they provide the only known general method which will take arbitrary definitions in a given logical form of sets of reals, and produce definitions of members of those sets." [37, p. 107].

We will not discuss Zermelo's original work, but rather a modern version of the argument that one of the players has a drawing strategy (a Gale-Stewart analysis via tree labelling), and point out how this can be seen to be an "*Anwendung der Mengenlehre*":

Since a position in the game of chess is determined by the distribution of 13 types of pieces[3] to 64 squares, we can also view chess positions as natural numbers between 0 and 13^{64}.[4] If you have two such natural numbers n and n^*, the rules of chess describe whether n^* can be reached from n in a single move: n^* has to be a position in which all but at most two pieces are in the same board position and the board position changes have to follow the rules of chess. Moreover, if n^* is reachable from n in a single move, we can uniquely determine the player that made the move from n to n^*. Let us write $\mathrm{Reach}_{\mathsf{WHITE}}(n^*, n)$ for "n^* is reachable from n by a single move of player WHITE" and $\mathrm{Reach}_{\mathsf{BLACK}}(n^*, n)$ for "n^* is reachable from n by a single move of player BLACK".

Look at sequences $S = \langle n_i\,;\, i \in \mathbb{N}\rangle$ of chess positions. If for all k we have either $\mathrm{Reach}_{\mathsf{WHITE}}(n_{k+1}, n_k)$ or $\mathrm{Reach}_{\mathsf{BLACK}}(n_{k+1}, n_k)$ depending on whether k is even or odd (we shall call this condition the *legality condition*), then S is an infinite run of a chess game in which no player violates any rules. Since every chess game that is won by either player ends in a violation of a chess rules (note that it is a chess rule that you may not move your King into a check position), we shall call those sequences a *Draw*. On the other hand, if we have a sequence such that for some k this condition is not met, we shall call it a *Win for* WHITE if the least such k is odd and a *Win for* BLACK if it is even.

That way, we partitioned the set of all infinite sequences of positions into three subsets: the set D of Draws, the set W of Wins for WHITE and the set B of Wins for BLACK.

In this mathematized setting for the game of chess, we can easily say what it means for player WHITE to have a drawing strategy: a drawing strategy is a function that assigns to each finite sequence

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[3]The types of pieces are: No piece, white and black King, white and black Queen, white and black Knight, white and black Bishop, white and black Rook, and white and black Pawn.

[4]Lots of these positions are not chess positions that can be achieved without rules violations, *e.g.*, every position that contains two Kings of the same colour *etc.* *Cf.* [1, p. 171–172] for a discussion of the real complexity of the game tree of chess.

$\langle n_0, \dots, n_{2k}\rangle$ of chess positions of even length that satisfies the legality condition a new position n_{2k+1} such that the extended sequence $\langle n_0, \dots, n_{2k+1}\rangle$ still satisfies the legality condition. Playing according to such a function (assuming it exists) will guarantee that –regardless of what BLACK plays– the result will either be an infinite sequence satisfying the legality condition (*i.e.*, a Draw) or a sequence in which BLACK is the first to violate the legality condition, hence an element of W. Similarly, we define a drawing strategy for BLACK and see that if BLACK follows such a strategy, the outcome will be either in D or in B.

Note that you can view the set P of finite sequences of positions as a graph $G = \langle V, E\rangle$ with $V := P$ and E is the set of $\langle p, p^*\rangle$ such that p^* is a extension of p by one position n^* such that n^* is reachable from the last position of p by a legal move of the appropriate player. This graph is actually a tree, and we can see strategies as subtrees of this tree of a very specific kind: A strategy for WHITE is a subtree that has a single successor for nodes of even depth and a full branching into all possible successors for nodes of odd depth (and dually for BLACK).

After this preparatory work, we can restate the determinacy of the game of chess in a precise way:

Theorem 1.1. *Either* WHITE *or* BLACK *has a drawing strategy.*

Proof. We look at the set P of all finite sequences of positions and colour this set with three colours, WHITE, GREY and BLACK, in a recursive process. Any finite sequence p of positions carries the information which player has to move next: if p has even length, WHITE has to move, if p has odd length, BLACK has to move.

If $p \in P$ is a finite sequence of positions that doesn't satisfy the legality condition, check who violates it first. If BLACK violates it first, colour p WHITE, if WHITE violates it first, colour p BLACK.

Suppose you have already coloured all elements of a subset $P^* \subsetneq P$. Take some $p \in P \setminus P^*$. If WHITE has to move from p, and p has the property that there is a position p^* with $\mathrm{Reach}_{\mathsf{WHITE}}(p^*, p)$ and p^* is already coloured WHITE, then we colour p WHITE as well. If, on the other hand, p has the property that for all positions p^* with $\mathrm{Reach}_{\mathsf{WHITE}}(p^*, p)$, p^* is already coloured BLACK, then we colour p BLACK as well. Then we colour positions $p \in P \setminus P^*$ such that BLACK has to move from p dually.

These rules describe a recursive process that allows to subsequently colour more and more positions in P, until you reach a point in the recursion where no $p \in P \setminus P^*$ satisfies the conditions to get a colour anymore. When you reach that point (a *fixed point* of the recursion), you stop the recursion and colour all remaining positions GREY.

This recursion is possibly transfinite: In general, if there are infinitely many possible moves in each step, it is easy to construct games such that you need more than ω steps to reach the fixed point of the recursion. This is where set theory enters the picture — the proof of the existence of a fixed point and the fact that transfinite recursions work are part of (elementary) set theory. (Note that since there are only at most 13^{64} chess positions, the tree for chess is finitely branching and therefore the recursion is not transfinite in this case: a standard recursion over natural numbers is enough.)

When the whole set P is coloured, it is easy to see that the colour of the empty sequence $\varnothing$ determines the value of the game.

Suppose the empty sequence is WHITE. We think of the elements of P labelled not only by WHITE, GREY, and BLACK, but also by the ordinal α_p of the stage of the transfinite recursion in which p was coloured.

If some sequence of positions p received the colour WHITE and it's of even length, then there is some immediate extension of p that received the colour WHITE at some earlier stage. If it received the colour WHITE and it's of odd length, then all immediate legal extensions must have received the colour WHITE at an earlier stage.

Thus, we can define a strategy for WHITE by demanding that if he has to play at p, he should play an extension q such that q was coloured at an earlier stage than p.

Clearly, this strategy makes sure that for each run, the sequence of ordinals α_p is a descending sequence of ordinals and each p in the run is coloured WHITE. Every descending sequence of ordinals hits 0 after finitely many steps. Thus the run of the game hits a sequence p that has been coloured WHITE at stage 0 of the recursion, *i.e.*, BLACK has violated some rule.

This is where set theory enters the proof: we are using the well-foundedness of the sequence of stages of the transfinite construction to prove that the strategy is winning.

Similarly, if the empty sequence is BLACK, BLACK has a winning strategy.

If the empty sequence is GREY, both players can guarantee that the sequence stays legal for infinitely many steps and thus create a Draw.

These three possibilities together prove the theorem. q.e.d.

The mathematization of a concrete game has succeeded in proving an abstract theorem. (Of course, so far without practical consequences, since the colour of the empty sequence is unknown to even the fastest computers.[5])

1.3. **Stochastic Games.** Stochastic games have been used in statistics for a long time.

Before we go into the discussion, let us clarify our usage of the terms "stochastic game" and "imperfect information game" in this paper. In the game theoretic literature, "stochastic game" normally means a game in which the players move simultaneously and in each round an additional random event influences the outcome that is known to both players (*e.g.*, games involving a roll of dice in each round)[6], whereas "imperfect information game" is understood (following Aumann and Maschler) to mean games in which players play simultaneously and are not completely informed about the whole game situation (*e.g.*, card games with undisclosed hands).

We, on the other hand, take "stochastic game" and "imperfect information game" to be terms of the most general kind. When we say "stochastic game" we just mean a game involving any form of randomization, when we say "imperfect information game" we mean any game in which players have to move without knowing the entire situation.[7]

One particular application of stochastic games is to regard a two-player game interpreted by imagining player I to be a statistician trying to set up his empirical experiment in an optimal way and player II to be nature trying to fool him. The analysis of stochastic

[5] *Cf.* [1, p. 171–172].

[6] This use of "stochastic game" follows Shapley [36].

[7] In this sense, our pBl-AD is a stochastic game axiom but no imperfect information game axiom, while our Bl-AD is both. (For definitions, *cf.* Section 2.)

games provides methods of defining optimality of tests and setups of experiments.[8]

As we said, the set theoretic analysis of infinite games in the tradition of the Cabal dealt mostly with perfect information games. In the 1960's, there were some investigations of the foundations of stochastic games by David Blackwell [3] who proved that if the payoff set of a certain game modelled with probability measures is a countable unions of closed sets, then one of the players can approximate an optimal strategy. Such a result corresponds to a proof of determinacy in the context of perfect information games; in fact, we shall call such a property "Blackwell Determinacy" later on (*cf.* Section 2).

Not much happened in the research of the foundational aspects of infinite stochastic games for almost three decades[9] while the applications of these games became more and more important.

Recently, Blackwell himself in an extended abstract [4] revived the interest in the foundational questions about games of this sort. In particular, he asked whether all Borel sets have the determinacy property described. Soon thereafter, Marco Vervoort, Tony Martin and Itay Neeman [41, 42, 25, 43, 27], solved some of these fundamental questions[10] and started to build a basic knowledge about the set theoretic behaviour of imperfect information games.

The interest in foundational questions about imperfect information games in general and stochastic games in particular seems not just to be a question of the foundations of mathematics. Theoretical Computer Science has used both the stochastic or randomized viewpoint and the game viewpoint very effectively over the last decades, and a thorough understanding of the connections between perfect information game theory and the imperfect information variants is essential for further progress in the analysis of the use of games in Computer Science: Most of the techniques from Computer Science for perfect information games break down when you apply them to more general games.

1.4. **Objective and Prerequisites of this Paper.** This paper is a survey and research announcement that does not attempt to provide a general development of imperfect information games. In the paper,

[8] *Cf.* [5] and [9].

[9] There were results extending Blackwell's result by Orkin [35] and Maitra and Sudderth [24].

[10] In particular, Martin in [25] solved Blackwell's question about Borel sets.

we are connecting a very special kind of imperfect information games (later called *Blackwell games*) to the vast knowledge of set theoretic structure that is gained from game theoretic axioms about perfect information games.

We shall be dealing with two axioms of Blackwell Determinacy pBl-AD and Bl-AD which are stochastic and imperfect information analogues of the Axiom of Determinacy AD.[11]

It is known that AD implies Bl-AD and that Bl-AD implies pBl-AD (Theorem 2.1), and Tony Martin [25] has conjectured that AD and Bl-AD are equivalent. Part of that conjecture is settled, as Martin, Neeman and Vervoort [27] have shown that pBl-AD, Bl-AD and AD have the same consistency strength[12], but the question of equivalence is still open. Equiconsistency of two theories does not imply that both theories have the same structural consequences for the set theoretic universe. Thus, it is a first and very important step towards proving Martin's conjecture to show that the powerful combinatorial and structure-theoretic consequences of AD hold under either pBl-AD or Bl-AD.

In this survey, we describe how to get the strong partition property of $\aleph_1$ and further consequences for the infinitary combinatorics of projective ordinals from Bl-AD, and an interesting definability hierarchy of sets of reals under the assumption of pBl-AD. Since we cannot assume that the reader is familiar with the concepts from the determinacy context, we shall give an introduction with many definitions. The reader interested in details and proofs should consult [23].

We shall be using the standard notation of set theory: ω is the set of natural numbers, $\omega^{<\omega}$ the set of finite sequences of natural numbers, and ω^ω the set of infinite sequences of natural numbers. If $x \in \omega^\omega$, we write $x{\restriction}n$ to denote the unique n-tuple that has the first n entries of x.

We shall be talking about *Jónsson cardinals*, *Rowbottom cardinals* and *measurable cardinals*. All of these are large cardinal notions,

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[11] Bl-AD has been introduced by Vervoort in [41].

[12] Two theories T and S are said to be *equiconsistent* if they prove the same consistency statements, *i.e.*, if the sets $\{U\,;\,T \vdash \mathrm{Cons}(U)\}$ and $\{U\,;\,S \vdash \mathrm{Cons}(U)\}$ are the same. By Gödel's incompleteness theorem, the consistency strength hierarchy is an interesting and non-trivial measure of logical strength for axiom systems.

ordered according to increasing logical (consistency) strength. More about these large cardinal notions can be found in any set theory textbook, *e.g.*, [12]. The same is true for descriptive complexity classes. We mention $\mathbf{\Sigma}^1_n$, $\mathbf{\Pi}^1_n$ and $\mathbf{\Delta}^1_n$ in this survey. Definitions and basic properties can be found in modern set theory textbooks.

2. DEFINITIONS

As we mentioned, Blackwell determinacy has its roots in the game theory of imperfect information games of statistics. A very simple prototype of this kind of games is the well-known game of *Scissors-Paper-Stone*: Two players simultaneously chose one of the three symbols SCISSORS, PAPER and STONE and evaluate their (random) outcome using the rules "SCISSORS cut PAPER", "PAPER wraps STONE" and "STONE blunts SCISSORS".

Compared to the tree representation of the set of finite sequences P in the chess example, the simultaneity of the moves eliminates both the chance of representing strategies as trees and the chance of the existence of a winning strategy: Player II doesn't have any knowledge of player I's move before he announces his move. In other words: He has *imperfect information* about the game situation.

If mathematized, this uncertainty can be expressed by different means: we can use the extensive game form (due to von Neumann, Morgenstern and Kuhn [21]) to keep as close to the game representation from Section 1.2, or we can use randomization, as we shall do in the following.

In this paper, we shall not actually be mainly dealing with stochastic games in full generality, not even with Blackwell's "Games with Slightly Imperfect Information". Instead we assign probabilities to moves in perfect information games, thereby moving from the game tree of moves to a game tree of probabilities.

Although these games are not imperfect information games in our sense, experience shows that they do seem to capture the (main part of the) set theoretic strength of imperfect information games.

We look at infinite games of the following type: we have two players, let's call them player I and player II, and a fixed payoff set $A \subseteq \omega^\omega$. The players play infinitely often natural numbers, player I begins with a_0, player II answers with a_1, then player I plays a_2 and so on. After infinitely many rounds of the game, they have produced a sequence $\langle a_i \,;\, i \in \omega \rangle \in \omega^\omega$ which they compare to the payoff set A.

If $\langle a_i \ ; \ i \in \omega \rangle \in A$, then player I wins, otherwise, player II wins. We shall be using the standard notation for infinite games: If $x \in \omega^\omega$ is the sequence of moves for player I and $y \in \omega^\omega$ is the sequence of moves for player II, we let $x * y$ be the sequence constructed by playing x against y, *i.e.*,

$$(x * y)(n) := \begin{cases} x(k) & \text{if } n = 2k, \\ y(k) & \text{if } n = 2k+1. \end{cases}$$

Conversely, if $x \in \omega^\omega$ is a run of a game, then we let x_{I} be the part played by player I and x_{II} be the part played by player II, *i.e.*, $x_{\mathrm{I}}(n) = x(2n)$ and $x_{\mathrm{II}}(n) = x(2n+1)$. We shall extend this notation to sets of reals $X \subseteq \omega^\omega$ in the obvious way: $X_{\mathrm{I}} := \{x_{\mathrm{I}} \ ; \ x \in X\}$ and $X_{\mathrm{II}} := \{x_{\mathrm{II}} \ ; \ x \in X\}$.

As usual in set theory, we shall be working on Baire space ω^ω with the standard topology generated by the basic open sets $[s] := \{x \in \omega^\omega \ ; \ s \subseteq x\}$ for finite sequences $s \in \omega^{<\omega}$.[13] It is well-known that this topological space is homeomorphic to the irrational numbers, so we feel safe to call its elements *real numbers*.

As mentioned, we cannot model strategies in imperfect information games by trees of positions. Instead, our stochastic games are normally modelled by assuming that the players have a tree of probability measures in mind according to which they randomize their moves. Let us denote by ω^{Even} the set of finite sequences of natural numbers of even length, by ω^{Odd} the set of such sequences of odd length, and by $\mathrm{Prob}(\omega)$ the set of probability measures on ω.

We call a function $\sigma : \omega^{\mathrm{Even}} \to \mathrm{Prob}(\omega)$ a *mixed strategy for player I* and a function $\sigma : \omega^{\mathrm{Odd}} \to \mathrm{Prob}(\omega)$ a *mixed strategy for player II.*

Let us describe two particularly interesting types of mixed strategies:

A mixed strategy is called *Blackwell strategy* if it doesn't depend on the moves in the same turn, *i.e.*, if s and t have the same longest even subsequence, then we have $\sigma(s) = \sigma(t)$. Blackwell strategies correspond to infinitely repeated games of the Scissors-Paper-Stone type: Since σ may use only information up to the last completed turn, it simulates that the moves of players I and II are revealed simultaneously. This is what Blackwell calls "Games with Slightly Imperfect Information" in his [4] and what we shall call *Blackwell Games.*

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[13] This is the same as the product topology of the discrete topology on ω.

A mixed strategy σ is called *pure* if for all $s \in \mathrm{dom}(\sigma)$ the measure $\sigma(s)$ is a Dirac measure, *i.e.*, there is a natural number n such that $\sigma(s)(\{n\}) = 1$.[14]

If σ and τ are strategies for player I and II, respectively, then they completely describe a play of the game between these two players: Player I chooses his first move a_0 according to the probability measure $\sigma(\langle\ \rangle)$, then player II looks at a_0, consults his strategy about the measure $\tau(\langle a_0 \rangle)$ and plays according to that probability measure.

Let

$$\nu(\sigma,\tau)(s) := \begin{cases} \sigma(s) & \text{if } \mathrm{lh}(s) \text{ is even, and} \\ \tau(s) & \text{if } \mathrm{lh}(s) \text{ is odd.} \end{cases}$$

Then for any $s \in \omega^{<\omega}$, we can define

$$\mu_{\sigma,\tau}([s]) := \prod_{i=0}^{\mathrm{lh}(s)-1} \nu(\sigma,\tau)(s{\restriction}i)(\{s_i\}).$$

This generates a Borel probability measure on ω^ω which can be seen as a measure of how well the strategies σ and τ performs against each other. If B is a Borel set, $\mu_{\sigma,\tau}(B)$ is interpreted as the probability that the result of the game ends up in the set B when player I randomizes according to σ and player II according to τ.

Note that if σ and τ are both pure, then $\mu_{\sigma,\tau}$ is a Dirac measure concentrated on the unique real that is the outcome of this game denoted by $\sigma * \tau$.

To see the working of mixed strategies in an example, let $A := \{x_n\,;\, n \in \omega\}$ be a countable set of elements of 2^ω with the property that the semidiagonal function $n \mapsto x_n(2n)$ is not computable. Player II clearly wins the game with payoff A by playing $1 - x_n(2n)$ in his nth move, but this winning strategy is not computable. On the other hand, player II can guarantee that he wins with probability 1 by following the computable randomized strategy that plays 0 with probability $\frac{1}{2}$ and 1 with probability $\frac{1}{2}$ in every step. If you call this strategy τ, then (regardless of what strategy σ player I follows) $\mu_{\sigma\tau}$ is an atomless measure and hence A has measure 0.

In a trade-off, Player II traded his guarantee to "win always" with a complicated strategy for a guarantee to "win with probability 1" with a simple strategy.

[14] This is a rather odd (but equivalent) way of stating the definition of a strategy in the sense of Section 1.2.

This example gives us the important notions of a winning strategy and a strongly optimal strategy: We call a pure strategy σ for player I (τ for player II) a *winning strategy* if for all pure counterstrategies τ (σ), we have that $\sigma * \tau \in A$ ($\sigma * \tau \notin A$). For any mixed strategy σ (for player I) or τ for player II we can define a measure for its quality (the *value of the strategy*) by

$$\mathrm{val}_{\mathrm{I}}^{A}(\sigma) := \inf\{\mu^{-}_{\sigma,\tau}(A)\,;\,\tau \text{ is mixed strategy for player II}\}, \text{ and}$$

$$\mathrm{val}_{\mathrm{II}}^{A}(\tau) := \sup\{\mu^{+}_{\sigma,\tau}(A)\,;\,\sigma \text{ is mixed strategy for player I}\}.$$ [15]

A strategy for player I is now called *strongly optimal for* A if $\mathrm{val}_{\mathrm{I}}^{A}(\sigma) = 1$, and and a strategy τ for player II is called *strongly optimal for* A if $\mathrm{val}_{\mathrm{II}}^{A}(\tau) = 0$.

It is not obvious that it is reasonable to expect that there are strongly optimal strategies. In order to see that we can (under the right set-theoretical assumptions), we have to use the Vervoort Strong Zero-One Law [43, Theorem 5.3.4] and a Transfer Theorem of the present author [22, Corollary 4.3].

If we restrict ourselves to Blackwell strategies σ and τ, we cannot expect that the values converge to either 0 or 1. But it is easy to see that if you define the *value sets* for player I and player II by

$$V_{\mathrm{I}}(A) := \{\mathrm{val}_{\mathrm{I}}^{A}(\sigma)\,;\,\sigma \text{ is a Blackwell strategy for player I}\}, \text{ and}$$

$$V_{\mathrm{II}}(A) := \{\mathrm{val}_{\mathrm{II}}^{A}(\tau)\,;\,\tau \text{ is a Blackwell strategy for player II}\},$$

then $V_{\mathrm{II}}(A)$ lies entirely above $V_{\mathrm{I}}(A)$ in the sense that for all $v \in V_{\mathrm{II}}(A)$ and $v^* \in V_{\mathrm{I}}(A)$ we have $v \geq v^*$.

If now these two sets $V_{\mathrm{I}}(A)$ and $V_{\mathrm{II}}(A)$ touch each other in a point p (depicted in Figure 1), then the outcome of the game is stochastically determined in the following sense: both players can approximate the outcome that player I wins with probability p. In this case, we call the payoff set *Blackwell determined*.

In the other case, when the sets $V_{\mathrm{I}}(A)$ and $V_{\mathrm{II}}(A)$ don't touch each other, then the interval between the supremum v^- of $V_{\mathrm{I}}(A)$ and the infimum v^+ of $V_{\mathrm{II}}(A)$ is an area of indeterminacy: Player I can bound his chance of winning from below by v^- and player II can bound player I's chance of winning from above by v^+, but since

[15] Here, μ^+ denotes outer measure and μ^- denotes inner measure with respect to μ in the usual sense of measure theory. If A is Borel, then $\mu^+(A) = \mu^-(A) = \mu(A)$ for Borel measures μ.

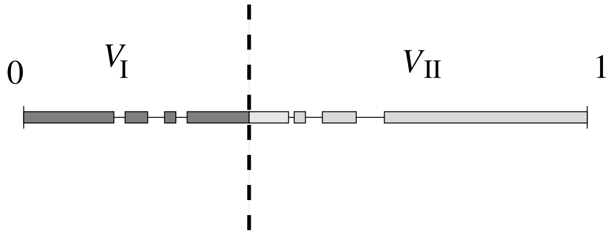

FIGURE 1. Value sets V_{I} and V_{II} touch each other: The outcome is stochastically determined if both players approximate optimal play

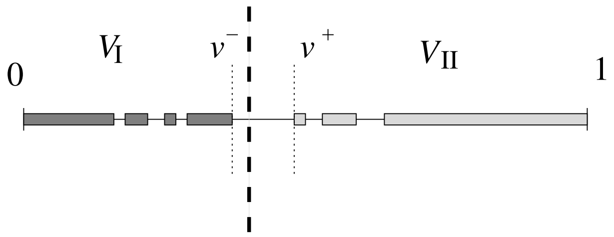

FIGURE 2. Value sets V_{I} and V_{II} don't touch: The payoff set is not Blackwell determined

these are not the same number, the real outcome can be somewhere in the interval (as depicted in Figure 2).

We are now in a position to define three different determinacy axioms:

The *Axiom of Determinacy* AD says: For every set A there is either a winning strategy for player I or a winning strategy for player II in the game with payoff set A.

The *Axiom of perfect information Blackwell Determinacy* pBl-AD says: For every set A there is either a strongly optimal strategy for player I or a strongly optimal strategy for player II in the game with payoff set A.

The *Axiom of Blackwell Determinacy* Bl-AD says: Every set $A \subseteq 2^\omega$ is Blackwell determined.[16]

What is the relationship between our three determinacy axioms?

Theorem 2.1. (a) AD *implies* Bl-AD, *and* (b) Bl-AD *implies* pBl-AD.

Proof. (a) is an instance of the main theorem of [25], and (b) is proved in [22] using the Vervoort Strong Zero-One Law [43, Theorem 5.3.4].

Instead of showing these two technically involved results, let us sketch the considerably easier concatenation of these two results: AD implies pBl-AD:[17]

For each mixed strategy τ for player II there is a probability measure v_τ on the set of (codes of) pure strategies with the property that for all Borel sets X,

$$\mu_{\sigma,\tau}(X) = \int \mu_{\sigma,\pi}(X)\ dv_\tau(\pi)$$

(and similarly for a mixed strategy for player I). This measure has been introduced by Marco Vervoort [41] and shall be called the *Vervoort code* of τ.

Let σ be a pure winning strategy for player I, then for all pure strategies π, $\mu_{\sigma,\pi}$ witnesses that player I wins with probability 1 against π. Let us now show that σ also wins against τ with probability 1.

Since we assume AD, all sets are universally measurable, so we can look at the function $\pi \mapsto \mu_{\sigma,\pi}(X)$ which is a constant function equal to 1. Thus, because of the defining property of the Vervoort code, this means that $\mu_{\sigma,\tau}$ must witness that player I also wins with probability 1 against τ. q.e.d.

As already mentioned, Tony Martin [25] conjectured that the converse of Theorem 2.1 (a) also holds.

Up to now, this question is not solved, but we shall get a rich structure theory of the ordinals up to $\aleph_{\omega+1}$ that is usually seen as

[16]It is important that Bl-AD is defined on subsets of 2^ω instead of ω^ω. It is easy to construct a very simple (clopen) subset of ω^ω that is not Blackwell determined: $\{x \in \omega^\omega\ ;\ x(0) \geq x(1)\}$. For more on this, *cf.* [22, Observation 2.3].

[17]Note that this implication is not trivial: a strongly optimal strategy has to be good against all mixed strategies, whereas a winning strategy only has to win against pure strategies.

being very characteristic of AD and thus can serve as an indication that Martin's conjecture is true.

3. CONSEQUENCES OF AD

In this section, we shall give the reader who is not an expert in descriptive set theory an overview of the rich theory that can be developed under the assumption of the Axiom of Determinacy. The reader who is interested in more details should read [12, §§27 & 28].

We have to start with a problem for the whole enterprise: Gale and Stewart [7, Theorem 1] have shown that the Axiom of Determinacy is incompatible with the Axiom of Choice AC, hence ZFC proves $\neg$AD.

This may be a seemingly insurmountable problem for true believers in the Axiom of Choice: why would they bother with an axiom that contradicts one of the more important axioms of set theory?

Yet, even those true believers in the Axiom of Choice investigate models of AD with a lot of interest. Why is this so?

Under certain richness assumptions for the universe we will always find very natural models of AD, so it makes a lot of sense to study AD because the world of functions and sets definable in real parameters might behave much more like an AD-model than like an AC-model (*cf.* [12, Corollary 32.14]):

Theorem 3.1 (Woodin). *Assume that there are ω Woodin cardinals and a measurable cardinal above them. Then* $\mathbf{L}(\mathbb{R}) \models \mathsf{ZF} + \mathsf{AD}$.

An even more pressing reason to develop the theory of AD than Theorem 3.1 is the fact that models of AD and knowledge about their structure and their properties figure prominently in all of higher set theory. As soon as the richness of the set theoretic universe under investigation is beyond a certain point, there will be an abundance of models of AD and their distribution in the hierarchy of models is connected with metamathematical properties of the universe. This is a point that cannot be discussed in this paper since it would involve a discussion of metamathematics beyond the horizon of this article.

In 1962, Mycielski and Steinhaus in their [32] started to investigate the Axiom of Determinacy (at that time still called "Axiom of Determinateness") because of its beautiful and strong consequences. The theory of AD has become one of the most intriguing parts of set theory, with surprising properties and an enormous amount of provable structure: Under AD, all sets are Lebesgue measurable, have the property of Baire, the perfect set property, and we even retain a large enough fragment of the Axiom of Choice (namely $\mathsf{AC}_\omega(\mathbb{R})$) to develop the basic theory of the real numbers and prove the standard theorems of analysis and topology.[18]

The basic theory for the rest of this survey will be $\mathsf{ZF}+\mathsf{DC}$ (the *Principle of Dependent Choices*).

3.1. **Infinitary Combinatorics.** The investigation of infinitary combinatorics under AD started in 1967, when Bob Solovay proved that the first uncountable cardinal $\aleph_1$ is a measurable cardinal under AD (again, this result violates the Axiom of Choice).

Let $\lambda < \kappa$ be regular cardinals. We define the *λ-closed unbounded filters* $\mathcal{C}_\kappa^\lambda$ on κ by

$$\mathcal{C}_\kappa^\lambda := \{X \subseteq \kappa \; ; \; \exists C \in \mathcal{C}_\kappa (C \cap \{\xi < \kappa \; ; \; \mathrm{cf}(\xi) = \lambda\} \subseteq X)\},$$

where $\mathcal{C}_\kappa$ is the usual closed unbounded filter on κ.

Definition 3.2. Let κ be a cardinal. We say that κ has the *strong partition property* if $\kappa \to (\kappa)^\kappa$ holds, and that κ has the *weak partition property*, if for all $\lambda < \kappa$, $\kappa \to (\kappa)^\lambda$ holds.

Both the weak and the strong partition property of any cardinal severely violate the Axiom of Choice, and they are extremely strong combinatorial properties.[19]

Even the weak partition property implies a very specific kind of measurability. Kleinberg has shown in [19], that if an uncountable κ has the weak partition property, then for every infinite regular $\lambda < \kappa$, the filter $\mathcal{C}_\kappa^\lambda$ is a κ-complete ultrafilter on κ.

In the late 60es and early 70es, Tony Martin, Robert Solovay and Ken Kunen were able to develop a rich theory of combinatorial structure on ω_1 under the assumption of AD. Of their theorems, the following two will be of most interest to us:

Theorem 3.3 (Solovay's Lemma). *Assume* AD. *Then for every $A \subseteq \omega_1$ there is a real $x \in \omega^\omega$ such that $A \in \mathbf{L}[x]$.*

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[18] *Cf.* [33, 30, 31, 6].

[19] Jim Henle called the strong partition property "one of the most powerful partition properties known to man [8, p. 151]". The Erdős arrow notation $\kappa \to (\lambda)^\mu$ means: For every colouring $F : [\kappa]^\mu \to 2$ there is a homogeneous set H of cardinality λ, *i.e.*, all elements of $[H]^\mu$ have the same F-colour. Here $[X]^\mu$ is the set of subsets of X with cardinality μ.

Theorem 3.4 (Martin)**.** *Assume* AD*. Then* $\aleph_1$ *has the strong partition property.*

Kleinberg was able to use the strong partition property in [20] to get very strong consequences. In order to describe Kleinberg's result, we need to use the notions of Jónsson cardinals, Rowbottom cardinals and measurable cardinals. As mentioned in Section 1.4, we shall not define these notions but refer the reader to [12].[20]

For the formulation of the following theorem we use the following convention: If U is a σ-complete ultrafilter on an ordinal α and β is another ordinal, then β^α/U is a well-ordered structure. We will identify this structure with the unique isomorphic ordinal γ, and write $\beta^\alpha/U = \gamma$.

Definition 3.5. Let κ be a cardinal with the strong partition property and μ a normal measure on κ. We then define a sequence of well-ordered structures $\langle \kappa_n^\mu \ ; \ n \leq \omega \rangle$ as follows:

- $\kappa_1^\mu := \kappa$,
- $\kappa_{n+1}^\mu := (\kappa_n^\mu)^\kappa/\mu$, and
- $\kappa_\omega^\mu := \sup\{\kappa_n^\mu \ ; \ n \in \omega\}$.

This sequence is called the *Kleinberg sequence derived from* μ.

Theorem 3.6 (Kleinberg)**.** *Let* κ *be a cardinal with the strong partition property and* U *be a normal ultrafilter on* κ*. Let* $\langle \kappa_i \ ; \ i \leq \omega \rangle$ *be the Kleinberg sequence derived from* U*. Then*

(a) *for all natural numbers* $n \in \omega$, $\kappa_n < \kappa_{n+1}$,
(b) κ_1 *and* κ_2 *are measurable,*
(c) *for all* $n \geq 2$, $\mathrm{cf}(\kappa_n) = \kappa_2$,

[20]For the purpose of the present paper, it is sufficient to know that the existence of Jónsson cardinals, Rowbottom cardinals or measurable cardinals is a strong richness assumption on the set theoretic universe. Their existence cannot be shown in ZFC and moreover, even the consistency of their existence cannot be shown in ZFC. (Statements like this are called *Large Cardinal Axioms* or *Strong Axioms of Infinity.*) In terms of consistency strength, the theory ZFC+"there is a measurable cardinal" is strictly stronger than ZFC+"there is a Rowbottom cardinal", this in turn is strictly stronger than ZFC+"there is a Jónsson cardinal", and this is strictly stronger than ZFC alone. The fact that some κ is measurable, Rowbottom or Jónsson tells us a lot about the combinatorics of κ.

The pattern of cardinals described in Corollary 3.7 is highly characteristic of AD-models in the sense that it is extremely hard to produce patterns like this from any other assumption, even those with very high consistency strength. For more about this, *cf.* [2].

(d) κ_n *is a Jónsson cardinal, and*
(e) $\sup\{\kappa_n\ ;\ n \in \omega\}$ *is a Rowbottom cardinal.*

Moreover, if $\kappa^\kappa/U = \kappa^+$, *then* $\kappa_{n+1} = (\kappa_n)^+$ *for all* $n \in \omega$.

Now Martin's Theorem 3.4 and Kleinberg's Theorem 3.6 give us immediately a second measurable κ_2 above $\aleph_1$ and a sequence of Jónsson cardinals beyond $\aleph_1$. It is natural to ask whether we can compute the exact position of these additional large cardinals. Using Solovay's Lemma 3.3, we can see that $\aleph_1$ and the measure $\mathcal{C}_{\omega_1}$ satisfy the "moreover" part of Kleinberg's Theorem 3.6.

Corollary 3.7. *Assume* AD. *Then* $\aleph_1$ *and* $\aleph_2$ *are measurable,* $\aleph_n$ *for* $3 \leq n < \omega$ *is Jónsson, and* $\aleph_\omega$ *is Rowbottom.*

The first uncountable cardinal $\aleph_1$ is not the only cardinal with the strong partition property under AD. We define the *projective ordinals* by

$$\boldsymbol{\delta}^1_n := \sup\{\xi\ ;\ \xi \text{ is the length of a prewellordering of } \omega^\omega \text{ in } \boldsymbol{\Delta}^1_n\}.$$

A lot was known about the projective ordinals in the 1970s (*cf.* [14]):

Fact 3.8. Let n be a natural number. Assume AD. Then:
(a) (Martin, Kunen 1971) all $\boldsymbol{\delta}^1_n$ are measurable,
(b) (Martin, Kunen 1971) $\boldsymbol{\delta}^1_1 = \aleph_1$, $\boldsymbol{\delta}^1_2 = \aleph_2$, $\boldsymbol{\delta}^1_3 = \aleph_{\omega+1}$, and $\boldsymbol{\delta}^1_4 = \aleph_{\omega+2}$,
(c) (Martin 1971) for all $\alpha < \omega_1$ the partition relation $\boldsymbol{\delta}^1_{2n+1} \to (\boldsymbol{\delta}^1_{2n+1})^\alpha$ holds,

Since the values of $\boldsymbol{\delta}^1_1$, $\boldsymbol{\delta}^1_2$, $\boldsymbol{\delta}^1_3$, and $\boldsymbol{\delta}^1_4$ were known, the next open question was the value of $\boldsymbol{\delta}^1_5$. This was the content of the First Victoria Delfino Problem and was solved by Steve Jackson who computed $\boldsymbol{\delta}^1_5$ (*cf.* [10] and [11]), and proved that all odd projective ordinals have the strong partition property, and all even projective ordinals have the weak partition property.

Theorem 3.9 (Jackson)**.** *Assume* AD. *Then*

$$\boldsymbol{\delta}^1_5 = \aleph_{\omega^{\omega^\omega}+1},$$

and $\boldsymbol{\delta}^1_3$ *has the strong partition property.*

Starting from this, Jackson developed his *description theory* that helped developing a combinatorial theory of the projective ordinals and even beyond.

3.2. **The Lipschitz Hierarchy.** Let A and B be subsets of ω^ω. We define the Lipschitz game as follows:[21]

Let

$$L_{A,B} := \{x \in \omega^\omega \ ; \ \neg(x_{\mathrm{I}} \in A \leftrightarrow x_{\mathrm{II}} \in B)\}.$$

The game with payoff set $L_{A,B}$ is called the *Lipschitz game for A and B*.

This game give rise to the relation of Lipschitz reducibility:

$$\begin{aligned} A \leq_\ell B \quad :\Longleftrightarrow \quad & \text{player II wins the Lipschitz game for } A \text{ and } B \\ \Longleftrightarrow \quad & \text{there is a Lipschitz function } f \text{ with } f^{-1}\text{''}\, B = A. \end{aligned}$$

As usual, the equivalence relation $\equiv_\ell$ is defined by

$$A \equiv_\ell B :\Longleftrightarrow A \leq_\ell B \ \& \ B \leq_\ell A.$$

The equivalence relations give rise to a degree structure. We call a set $\mathbf{d}$ of subsets of ω^ω *Lipschitz degree* if it is closed under $\equiv_\ell$.

If $\mathbf{d}$ is a Lipschitz degree, then we shall call

$$\breve{\mathbf{d}} := \{B \ ; \ \exists A \in \mathbf{d}(B \equiv_\ell \omega^\omega \backslash A)\}$$

the *dual of* $\mathbf{d}$. We call $\mathbf{d}$ *selfdual* if $\mathbf{d} = \breve{\mathbf{d}}$, *i.e.*, $[A]_\ell$ is selfdual if $A \equiv_\ell \omega^\omega \backslash A$.

We write $\bigoplus_{i\in\omega} A_i$ for $\bigcup_{n\in\omega} \langle n \rangle^\frown A_n$.

The work of William Wadge, Tony Martin, Donald Monk, John Steel and Robert Van Wesep has established that the Lipschitz degrees exhibit a very characteristic global structure theory under the Axiom of Determinacy (again, violating the Axiom of Choice). These results form one of the most fundamental framework for Cabal style Descriptive Set Theory.

The most prominent of these properties are:

Fact 3.10. Assume AD. Then the Lipschitz hierarchy has the following properties:

(a) The Lipschitz degree are semi-linearly ordered, *i.e.*, for every two degrees $\mathbf{d}$ and $\mathbf{e}$, we either have $\mathbf{d} \leq_\ell \mathbf{e}$, or $\mathbf{e} \leq_\ell \mathbf{d}$, or the set of $<_\ell$ predecessors and $<_\ell$ successors of $\mathbf{d}$ and $\mathbf{e}$ coincide and $\mathbf{d} = \breve{\mathbf{e}}$.

[21] The Lipschitz game was introduced by William Wadge [44, 40].

(b) The relation $\leq_\ell$ is wellfounded. Together with the semi-linear ordering principle this means that we can assign an ordinal rank to each Lipschitz degree. (A non-selfdual degree and its dual get the same ordinal rank.)

(c) The Lipschitz degrees whose ranks are of uncountable cofinality are nonselfdual, all others (except for the first two degrees) are selfdual.

4. The Results and Beyond

We have been able to show most of the prominent and characteristic features of AD that we listed in Section 3 under the weaker assumption of Bl-AD or even pBl-AD. These results are (apart from an early result about Lebesgue measurability by Vervoort [41]) the first known consequences of Bl-AD that contradict the Axiom of Choice and that give a global structure to the family of sets of real numbers.

Why is it hard to transfer the proofs of these results from an AD setting to an Bl-AD setting?

The main problem in proving these results is that a winning strategy σ (say, for player I) gives us a function $f_\sigma : \omega^\omega \to \omega^\omega$ that guarantees that for all $x \in \omega^\omega$, we have $f_\sigma(x) \in A$.[22] On the other hand, a strongly optimal strategy σ gives only a function assigning to x a measure $\mu_{\sigma,x}$ that gives the set A measure one, but we seem to be lacking a way of picking an element of A using that measure as input data.[23]

In a special type of games in which a prewellordering is used to evaluate the quality of the moves of both players (we shall call them *games using boundedness*), we can replace the use of the function f_σ in a way that still works in the Blackwell context. We call this technique the *simulation technique*. The simulation technique will in general work under the assumption of pBl-AD (and doesn't need Bl-AD).

Let $A \subseteq \omega^\omega$ be a set of reals and $\leq$ any prewellordering of ω^ω. Also fix a mixed strategy σ for player I and a mixed strategy τ for player II.

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[22]By $f_\sigma(x) := \sigma * \tau$ where τ is the trivial strategy playing x digit by digit regardless of the moves of player I.

[23]Compare this to Steel's quote about tree representations in Footnote 2.

Let $U_x^{\leq,\mathrm{I}} := \{u \ ;\ x_{\mathrm{I}} \leq u_{\mathrm{I}}\}$ and $U_x^{\leq,\mathrm{II}} := \{u \ ;\ x_{\mathrm{II}} \leq u_{\mathrm{II}}\}$. Using this notation, we define the *$\leq$-pseudoimage of A under σ* (under τ):

$$\Psi_{\leq}^{\sigma,\mathrm{I}}(A) := \left\{x_{\mathrm{I}} \ ;\ \exists z \in A \left(\mu^-_{\sigma,z}(U_x^{\leq,\mathrm{I}}) > 0\right)\right\}, \text{ and}$$

$$\Psi_{\leq}^{\tau,\mathrm{II}}(A) := \left\{x_{\mathrm{II}} \ ;\ \exists z \in A \left(\mu^+_{z,\tau}(U_x^{\leq,\mathrm{II}}) > 0\right)\right\}.$$

If σ is a pure strategy, then the pseudoimage is just the closure of the usual image (of the projection of f_σ to the first component) under $\geq$. Suppose (as is true in our special cases) that our payoff set A has the property

$$\text{if } x \in A \text{ and } x_{\mathrm{I}} \leq z, \text{ then } z * x_{\mathrm{II}} \in A.$$

Then the pseudoimage of any nonempty set under a pure winning strategy is contained in A.

Some weak version of this is retained for strongly optimal strategies:

Proposition 4.1. *Let $\leq$ be a prewellordering of ω^ω, and Y an arbitrary nonempty set of reals.*

(i) *If σ is a strongly optimal strategy for player I for the set A, then*

$$\Psi_{\leq}^{\sigma,\mathrm{I}}(Y) \cap \{x_{\mathrm{I}} \ ;\ x \in A, x_{\mathrm{II}} \in Y\} \neq \emptyset.$$

(ii) *If τ is a strongly optimal strategy for player II for the set A, then*

$$\Psi_{\leq}^{\tau,\mathrm{II}}(Y) \cap \{x_{\mathrm{II}} \ ;\ x \in A, x_{\mathrm{I}} \in Y\} \neq \emptyset.$$

Proof. For notational simplicity, we shall just show (i).

Let $y \in Y$. Since σ is strongly optimal, we know that

$$\begin{aligned} 1 &= \mu^-_{\sigma,y}(\{x_{\mathrm{I}} \ ;\ x \in A, x_{\mathrm{II}} \in Y\}) \\ &= \mu^-_{\sigma,y}(A). \end{aligned}$$

Let now $x_{\mathrm{I}} \in \{z_{\mathrm{I}} \ ;\ z \in A, z_{\mathrm{II}} \in Y\}$ with such that the $\leq$-rank of x_{I} is minimal. Clearly,

$$\mu^-_{\sigma,y}(U_x^{\leq,\mathrm{I}}) = 1,$$

and so $x_{\mathrm{I}} \in \Psi_{\leq}^{\sigma,\mathrm{I}}(Y)$. q.e.d.

We will now show an example of a use of the pseudoimage. Note that the statement of the following observation follows trivially from the fact that in this game, player I cannot have a pure winning strategy and the proof of Theorem 2.1. Nevertheless, we'll prove it here for expository reasons.

Fix a bijection $\beta : \omega^2 \to \omega$. For any element of Baire space, $x \in \omega^\omega$, we can define a relation R_x on the natural numbers by

$$nR_x m \iff x(\beta(n,m)) = 1.$$

Let WO be the set of codes of countable ordinals, *i.e.*, the set of $x \in \omega^\omega$ such that R_x is a wellorder. For $x \in \mathrm{WO}$, let $\|x\|$ be the order type of $\langle \omega, R_x \rangle$.

Observation 4.2. *Define the set A as follows:*

$$x \in A :\iff x_{\mathrm{I}} \in \mathrm{WO} \ \& \ (x_{\mathrm{II}} \notin \mathrm{WO} \vee \|x_{\mathrm{I}}\| > \|x_{\mathrm{II}}\|).$$

Then player I cannot have a strongly optimal strategy.

Proof. If σ is any strongly optimal strategy for player I, then $\Psi^{\sigma,\mathrm{I}}_{\leq}(\omega^\omega)$ is a $\boldsymbol{\Sigma}^1_1$ subset of WO (this uses a theorem of Tanaka [13]). By the standard Boundedness Lemma [12, Proposition 13.4], there is an ordinal $\alpha < \omega_1$ such that $\|y\| < \alpha$ for all reals $y \in \Psi^{\sigma,\mathrm{I}}_{\leq}(\omega^\omega)$. But then let player II play a code w for α. By Proposition 4.1,

$$\Psi^{\sigma,\mathrm{I}}_{\leq}(\{w\}) \cap \{x_{\mathrm{I}} \ ; \ x \in A, x_{\mathrm{II}} = w\} \neq \emptyset,$$

but this is impossible since for all $y \in \Psi^{\sigma,\mathrm{I}}_{\leq}(\{w\})$, we have $\|y\| < \|w\|$, so $y * w \notin A$. q.e.d.

The proof of Observation 4.2 is an extremely simple but generic example of how the simulating technique works in games where we have a natural prewellordering evaluating which player wins. Using these techniques, we can prove the following theorems:

Theorem 4.3. *Assume* Bl-AD. *Then $\aleph_1$ has the strong partition property, and for all natural numbers n, the odd projective ordinals $\boldsymbol{\delta}^1_{2n+1}$ have the countable partition property: for all $\alpha < \omega_1$, the partition relation $\boldsymbol{\delta}^1_{2n+1} \to (\boldsymbol{\delta}^1_{2n+1})^\alpha$ holds.*[24]

Theorem 4.4. *Assume* pBl-AD. *Then for every $A \subseteq \omega_1$ there is a real $x \in \omega^\omega$ such that $A \in \mathbf{L}[x]$.*

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[24]The countable partition relation for odd projective ordinals is provable under the assumption of pBl-AD. The proof of the strong partition property of $\aleph_1$ also works under the assumption of pBl-AD except for a use of "All sets of reals are Lebesgue measurable" of which we don't know how to prove it in ZF + pBl-AD.

Theorem 4.3, Theorem 4.4, and Kleinberg's Theorem 3.6 allow us to deduce:

Corollary 4.5. *Assume* $\mathsf{Bl\text{-}AD}$*. Then* $\aleph_1$ *and* $\aleph_2$ *are measurable,* $\aleph_n$ *(for* $3 \leq n < \omega$*) is Jónsson and has cofinality* $\aleph_2$*, and* $\aleph_\omega$ *is Rowbottom.*

That the cardinals between $\aleph_2$ and $\aleph_{\omega+1}$ are singular is important, because we can use that and Theorem 4.3 to compute $\boldsymbol{\delta}^1_3$ to be $\aleph_{\omega+1}$: we have $\boldsymbol{\delta}^1_3 \leq \aleph_{\omega+1}$ and since $\boldsymbol{\delta}^1_3 > \boldsymbol{\delta}^1_2 = \aleph_2$ is regular by virtue of Theorem 4.3, $\aleph_{\omega+1}$ is the only possible choice left for $\boldsymbol{\delta}^1_3$.

In addition to that, we can define a *Blackwell Lipschitz hierarchy* by setting

$$A \leq_{\mathrm{Bl}\ell} B :\Longleftrightarrow \text{ player II has a strongly optimal strategy for } L_{A,B},$$

defining a notion of Blackwell Lipschitz degree $[A]_{\mathrm{Bl}\ell}$ as above, and prove the following result about the derived degree structure:

Theorem 4.6. *Assume* $\mathsf{pBl\text{-}AD}$*. Say that a Blackwell Lipschitz degree* $\mathbf{d}$ *is called a* successor degree *if there is a degree* $\mathbf{p} <_{\mathrm{Bl}\ell} \mathbf{d}$ *such that there is no* $\mathbf{e}$ *with* $\mathbf{p} <_{\mathrm{Bl}\ell} \mathbf{e} <_{\mathrm{Bl}\ell} \mathbf{d}$*. We say that* $\mathbf{d}$ *is* of countable cofinality *if there is a sequence* $\langle [A_n]_{\mathrm{Bl}\ell}\ ;\ n \in \omega\rangle$ *without a greatest element such that*

$$A \equiv_{\mathrm{Bl}\ell} \bigoplus_{n\in\omega} A_n.$$

Then the Blackwell Lipschitz degrees are semi-linearly ordered, and every successor degree is a selfdual degree and the nonselfdual degrees are exactly the first two degrees and the nonsuccessor degrees which are not of countable cofinality.

Together, Theorems 4.3, 4.4, and 4.6 show the analogues of Theorem 3.3, Theorem 3.4, (5) of Fact 3.8, and (1) and (3) of Fact 3.10. Most of these properties are very characteristic for AD-situations and can be counted as evidence that $\mathsf{Bl\text{-}AD}$ might imply AD.

But there are many things left open: We couldn't show most of the general theory of the projective ordinals (Fact 3.8), in particular the measurability of the even projective ordinals and the computation of $\boldsymbol{\delta}^1_4$. What seems to be missing here is a version of the Moschovakis Coding Lemma [29, 7D.5] that allows us to view subsets of $\boldsymbol{\delta}^1_n$ as sets of reals of a tractable complexity. The proof of the Moschovakis Coding Lemma uses games which can not be dealt with by a use of the simulation technique.

Likewise, the wellfoundedness of the Blackwell Lipschitz hierarchy has not yet been shown.

As described earlier, the main problem with simulating proofs under pBl-AD is that there is no way to pick winning outcomes from the assignment of a measure that guarantees a win with probability 1. In other words: We are missing a principle that allows a parametrised choice of an element of the payoff set A given that the opponent follows a fixed real. In unpublished work (2001), the present author has identified such a principle called the *Parametrised Choice Principle* PCP of which he could show that (in the theory $\mathsf{ZF} + \mathsf{DC} + \mathsf{pBl\text{-}AD}$) PCP and AD are equivalent.

Thus, PCP can be seen as the difference between AD and pBl-AD. If we could show PCP from pBl-AD, Martin's conjecture would be proved.

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