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An Extension of a Commutativity Theorem of M. Uchiyama

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ABSTRACT. M. Uchiyama identified a necessary and sufficient condition for two nonnegative bounded operators to commute. We give a sufficient condition which apparently requires less.

Throughout, let X be a Hilbert space. By $\mathfrak{L}(X)$ we denote the bounded linear operators on X, and by $\mathfrak{L}^{\mathrm{sa}}(X)$ the (real) subspace of selfadjoint operators. We order $\mathfrak{L}^{\mathrm{sa}}(X)$ by the usual cone of nonnegative (definite) operators. In [5], M. Uchiyama established the following result.

Theorem 1. *Two nonnegative operators $A_1, A_2 \in \mathfrak{L}(X)$ commute if and only if for $n = 1, 2, \ldots$*

$$A_1^n A_2 + A_2 A_1^n \geq 0. \tag{1}$$

The condition (1) is of some interest, because it can be written in terms of the Weyl calculus. Recall that the Weyl calculus is a way of forming functions of several (not necessarily commuting) selfadjoint operators [1, 4]. For the particular case of operators $A_1, A_2 \in \mathfrak{L}^{\mathrm{sa}}(X)$ the Weyl calculus for the pair $\boldsymbol{A} := (A_1, A_2)$ assigns to the monomial $p(x_1, x_2) := x_1^n x_2^k$ the operator

$$W_{\boldsymbol{A}}(p) = \binom{n+k}{n}^{-1} \sum_{\pi} A_{\pi(1)} \cdots A_{\pi(n+k)},$$

where the sum is taken over all functions $\pi \colon \{1, \ldots, n+k\} \to \{1, 2\}$ which attain the value 1 precisely n times (there are $\binom{n+k}{n}$ such functions), i.e. the calculus "symmetrices" polynomials. For the particular monomials p_n of degree n which are defined by $p_0(x_0, x_1) := 1$

and $p_n(x_1, x_2) := x_1^{n-1} x_2$, the above formula becomes

$$W_{\boldsymbol{A}}(p_n) = \frac{1}{n} \sum_{k=0}^{n-1} A_1^k A_2 A_1^{n-1-k},$$

from which it follows that

$$(n+1) W_{\boldsymbol{A}}(p_{n+1}) - (n-1) A_1 W_{\boldsymbol{A}}(p_{n-1}) A_1 = A_1^n A_2 + A_2 A_1^n$$

or, similarly, that

$$2(n+1) W_{\boldsymbol{A}}(p_{n+1}) - n A_1 W_{\boldsymbol{A}}(p_n) - n W_{\boldsymbol{A}}(p_n) A_1 = A_1^n A_2 + A_2 A_1^n.$$

These identities express the left-hand-side of (1) in terms of $W_{\boldsymbol{A}}(p_n)$ and A_1. Note that the operators $W_{\boldsymbol{A}}(p_n)$ are all selfadjoint (this follows from the above formulae and is also a general property of the Weyl calculus of a real function [1]).

In a similar manner, one can write down other recursion formulae for $W_{\boldsymbol{A}}(p_n)$ which, by insertion into the above formulae, give many other expressions for $A_1^n A_2 + A_2 A_1^n$ in terms of $W_{\boldsymbol{A}}(p_n)$ and A_1. A typical sample of a commutativity result in terms of the Weyl calculus (using the above formulae and Theorem 1) is as follows.

Corollary 1. *Let $A_1, A_2 \in \mathfrak{L}(X)$ be nonnegative. Then $A_1 A_2 = A_2 A_1$ if and only if*

$$W_{\boldsymbol{A}}(p_{n+1}) \geq \frac{n-1}{n+1} A_1 W_{\boldsymbol{A}}(p_{n-1}) A_1, \qquad (n = 1, 2, \ldots).$$

The main interest of (1) is that it implies $A_1 A_2 = A_2 A_1$. We intend to prove commutativity in the situation that one does not know (1) a priori in full strength or perhaps not for all indices n. Of course, as a conclusion one then obtains that (1) actually holds for all $n = 1, 2, \ldots$. Let us first give an example which shows, even if X is finite dimensional, that it is not sufficient to verify (1) for any fixed finite family of indices n.

Example 1. Consider $X := \mathbb{C}^2$ and the noncommuting nonnegative matrices $A_1 := \left(\begin{smallmatrix} c & 0 \\ 0 & 1 \end{smallmatrix}\right)$ $(1 \neq c \geq 0)$ and $A_2 := \left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$. Since the $(1,1)$-entry of $A_1^n A_2 + A_2 A_1^n = \left(\begin{smallmatrix} 4c^n & c^n+1 \\ c^n+1 & 2 \end{smallmatrix}\right)$ is nonnegative for every n, this matrix is nonnegative if and only if its determinant $-(c^n)^2 + 6c^n - 1$ is nonnegative. Clearly, for any N, there exists $c \neq 1$ but "close to 1" such that $A_1^n A_2 + A_2 A_1^n \geq 0$ for $n = 1, \ldots, N$, although $A_1 A_2 \neq A_2 A_1$.

Example 1 demonstrates impressively that Theorem 1 above is an "asymptotic" result: The "closer" the operators are to commuting (i.e. the closer c is to 1), the larger n must be chosen so that $A_1^n A_2 + A_2 A_1^n \not\geq 0$.

To obtain the announced generalization of Theorem 1, we need the associativity of the standard calculus for bounded normal operators. This property is probably known; since we could not find a reference, a short proof is included.

Proposition 1. *Let $A \in \mathfrak{L}(X)$ be normal and $\sigma(A)$ denote the spectrum of A. If $f\colon \sigma(A) \to \mathbb{C}$ and $g\colon \sigma(f(A)) \to \mathbb{C}$ are bounded Borel functions, then the standard calculus for normal operators satisfies*

$$g(f(A)) = (g \circ f)(A). \tag{2}$$

Before we turn to the proof, a comment on the right-hand-side of (2) is in order. Since only $\sigma(f(A)) \subseteq \overline{f(\sigma(A))}$ (see e.g. [2, Section X.2, Corollary 9]), it is not the case, in general, that $g \circ f$ is defined on $\sigma(A)$. However, the set where it is not defined is only a null set with respect to the resolution of identity of A, i.e. the right-hand-side of (2) is well-defined by extending g to an arbitrary Borel function on a Borel set containing $f(\sigma(A))$, if necessary.

Proof. Let P denote the resolution of the identity of A. Then $f(A) = \int_{\sigma(A)} f\, dP$ and $\operatorname{supp}(P) = \sigma(A)$; to see this let f be the identity function on $\sigma(A)$ in [2, Section X.2, Corollary 9(iii)]. The resolution of the identity P_f of $f(A)$ is given by $P_f(B) := P(f^{-1}(B))$, see e.g. [2, Section X.2, Corollary 10]. Analogously, the resolution of the identity of $g(f(A))$ is then given by $(P_f)_g$, where $(P_f)_g(B) := P(f^{-1}(g^{-1}(B))) = P((g \circ f)^{-1}(B)) =: P_{g \circ f}(B)$. The same result holds, of course, for any extension of g to a bounded Borel function on a Borel set containing $f(\sigma(A))$. Applying [2, Section X.2, Corollary 10] to $\widetilde{g} \circ f$, where $\widetilde{g}$ denotes another such extension, we find that the resolution of the identity of $\widetilde{g} \circ f$ is given by $P_{\widetilde{g} \circ f} = (P_f)_{\widetilde{g}} = (P_f)_g$ (in particular, $(\widetilde{g} \circ f)(A) = \int_{\sigma(A)} id\; d(P_f)_g = g(f(A))$). Accordingly, $\widetilde{g}(f(A))$ is independent of the choice of the extension $\widetilde{g}$, and so $\widetilde{g} \circ f = g \circ f$ except possibly on a P-null set and thus, $(g \circ f)(A)$ is defined and equal to $(\widetilde{g} \circ f)(A) = g(f(A))$. $\square$

If we combine Proposition 1 with the fact that the inverse of any Borel function is automatically a Borel function (because it maps Borel sets onto Borel sets [3, §39, V, Theorem 1]), we obtain:

Proposition 2. *Let $A \in \mathfrak{L}(X)$ be normal. If $f\colon \sigma(A) \to \mathbb{C}$ is an injective, bounded Borel function, then an operator $B \in \mathfrak{L}(X)$ commutes with A if and only if it commutes with $f(A)$.*

Proof. Let f^{-1} denote a left-inverse (extended to a Borel function on $\overline{f(\sigma(A))}$). Recall that any bounded operator that commutes with A also commutes with $f(A)$, because it commutes with each spectral projection. Hence, if $B \in \mathfrak{L}(X)$ commutes with $f(A)$, then it also commutes, in view of Proposition 1, with $g(f(A)) = (g \circ f)(A) = id(A) = A$. $\square$

By a fractional power A^α of a nonnegative operator $A \in \mathfrak{L}(X)$, we mean the operator $f(A)$ with $f(x) = x^\alpha$ $(\alpha > 0)$ in the sense of the standard calculus. Note that this is the "right" definition for the power, since Proposition 1 implies $(A^\alpha)^\beta = A^{\alpha+\beta}$.

Corollary 2. *Let $A_1, A_2 \in \mathfrak{L}(X)$ be nonnegative. Then $A_1A_2 = A_2A_1$ if and only if $A_1^\alpha A_2^\beta = A_2^\beta A_1^\alpha$ for some, and thus all, (not necessarily integer) powers $\alpha, \beta > 0$.*

For an operator $A \in \mathfrak{L}^{\mathrm{sa}}(X)$, it will be convenient to use the notation

$$\min A := \min \sigma(A) = \inf_{\|x\|=1} \langle Ax, x\rangle \,.$$

We are now in a position to formulate the main result.

Theorem 2. *Two nonnegative operators $A_1, A_2 \in \mathfrak{L}(X)$ commute if and only if there are constants $\varepsilon > 0$, $j \in \mathbb{N}$, $\ell > 0$, and $c, d \geq 0$ such that, for all $n = j, 2j, 3j, \ldots$ we have*

$$(d+2\min A_2)c^n + \sum_{k=1}^{n} \min\left(A_1^{\ell\cdot k}A_2 + A_2A_1^{\ell\cdot k} + dA_1^{\ell\cdot k}\right)\binom{n}{k}\varepsilon^k c^{n-k} \geq 0. \tag{3}$$

In this case, each summand in (3) *is actually nonnegative for each choice of the constants $j \in \mathbb{N}$, $\ell > 0$, and $c, d \geq 0$.*

Proof. If $A_1A_2 = A_2A_1$, then $\min(A_1^\alpha A_2 + A_2A_1^\alpha) \geq 0$ for any $\alpha > 0$, and so (3) follows for any choice of the constants. Conversely, assume that (3) holds. Put $B_1 := \varepsilon A_1^\ell + cI$ and $B_2 := A_2 + \frac{d}{2}I$. Then,

$$
\begin{aligned}
(B_1^j)^n B_2 + B_2 (B_1^j)^n &= B_1^{jn} B_2 + B_2 B_1^{jn} \\
&= \sum_{k=0}^{jn} \binom{jn}{k} \left(A_1^{\ell\cdot k} A_2 + A_2 A_1^{\ell\cdot k} + d A_1^{\ell\cdot k} \right) \varepsilon^k c^{jn-k} \\
&\geq (dI + 2A_2) c^{jn} \\
&\quad + \sum_{k=1}^{jn} \binom{jn}{k} \min \left(A_1^{\ell\cdot k} A_2 + A_2 A_1^{\ell\cdot k} + d A_1^{\ell\cdot k} \right) I \varepsilon^k c^{jn-k} \\
&\geq 0 \quad \text{for all } n.
\end{aligned}
$$

Theorem 1 then implies $B_1^j B_2 = B_2 B_1^j$ which, in view of Corollary 2, is equivalent to $B_1 B_2 = B_2 B_1$ and thus to $A_1^\ell A_2 = A_2 A_1^\ell$ or to $A_1 A_2 = A_2 A_1$. □

Corollary 3. *Let $A_1, A_2 \in \mathfrak{L}(X)$ be nonnegative. Then $A_1 A_2 = A_2 A_1$ if and only if there exist $N \in \mathbb{N}$ and $d \geq 0$ such that the selfadjoint operators*

$$A_1^n A_2 + A_2 A_1^n + d A_1^n = (n+1) W_{\boldsymbol{A}}(p_{n+1}) - (n-1) A_1 W_{\boldsymbol{A}}(p_{n-1}) A_1 + d A_1^n,$$

for $n = N, 2N, 3N, \ldots,$ are nonnegative. In particular, $A_1 A_2 = A_2 A_1$ if (1) holds for all except possibly finitely many numbers n.

Proof. Apply Theorem 2 with $\ell = N$. □

Note, for $\|A_1\| \leq 1$ (which one can arrange by appropriate scaling), that the numbers $\min\left(A_1^\alpha A_2 + A_2 A_1^\alpha + d A_1^\alpha\right)$ which occur in (3) always have a lower bound which is independent of α. However, it appears that by applying only this fact and straightforward estimates to (3), one cannot generalize Corollary 3 to the situation when one does not have a priori information about these numbers in some infinite arithmetic sequence of α's. However, in finite dimensional spaces one can use a different argument to obtain commutativity.

Theorem 3. *Let X be finite dimensional, and let $A_1, A_2 \in \mathfrak{L}(X)$ be nonnegative. Then $A_1 A_2 = A_2 A_1$ if and only if there is some $d \geq 0$ and an unbounded subset $S \subseteq [0, \infty)$ such that*

$$A_1^\alpha A_2 + A_2 A_1^\alpha + d A_1^\alpha \geq 0 \qquad (\alpha \in S). \tag{4}$$

In this case, (4) actually holds for $S = [0, \infty)$ and each $d \geq 0$.

Proof. By considering an appropriate basis in X, we may assume that A_1 is represented by a diagonal matrix. Then the entries of the matrix $M(\alpha) := A_1^\alpha A_2 + A_2 A_1^\alpha + dA_1^\alpha$ are sums of terms of the form $c^\alpha d$ where $c > 0$ and $d \in \mathbb{R}$ are independent of α. Recall that $M(\alpha) \geq 0$ if and only if all the minors of $M(\alpha)$ are nonnegative. But, the minors of $M(\alpha)$ are sums (or differences) of products of entries of the matrix $M(\alpha)$, i.e., they have the form (after elementary manipulations) $\sum_{k=1}^K c_k d_k^\alpha$ with $0 < d_1 < \cdots < d_K$ and $c_k \neq 0$ independent of α. If $d_K \geq 1$, then the sign of this expression is the sign of c_K for all sufficiently large α. If $d_K < 1$, then the sign of this expression attains the sign of c_1 for all sufficiently large α. In all cases, the minors of $M(\alpha)$ do not change their sign for sufficiently large α, i.e. $M(\alpha_n) \geq 0$ for some sequence $\alpha_n \to \infty$ if and only if $M(\alpha) \geq 0$ for all sufficiently large α. Hence, the statement follows from Theorem 2. □

We do not know whether Theorem 3 also holds in general Hilbert spaces X. At least, there cannot be a counterexample which consists of an (infinite) diagonal matrix A_1 and a "block matrix" A_2.

Corollary 4. *Let $A_1, A_2 \in \mathfrak{L}(X)$ be nonnegative. Assume that there is some finite dimensional subspace $U \subseteq X$ which is invariant under A_1 and A_2 and such that the restrictions satisfy $A_1 A_2|_U \neq A_2 A_1|_U$. Then, for any $d \geq 0$ and any unbounded $S \subseteq [0, \infty)$, the relation* (4) *fails.*

Proof. For each sufficiently large $\alpha > 0$, the restriction of $B_\alpha := A_1^\alpha A_2 + A_2 A_1^\alpha + dA_1^\alpha$ to U fails to be nonnegative by Theorem 3. So, the extension B_α cannot be nonnegative either. □

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