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On Lie Derivations of 3-Graded Algebras

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Abstract. We prove that any graded Lie derivation on certain 3-graded associative algebras is a graded derivation. As an application we show that the graded Lie derivations on infinite dimensional topologically simple 3-graded associative H^*-algebras are also graded derivations.

1. Introduction

Over the years, there has been considerable effort made and success in studying the structure of derivations and Lie derivations of rings ([2, 3]), and Banach algebras ([12, 15, 16]). The 3-graded algebras have been considered in the literature with emphasis on their connections with Jordan pairs and the associated groups ([13, 14, 17]), we have also introduced in [5] techniques of derivations and 3-graded algebras in the treatment of problems of Lie isomorphisms. We are interested in investigating the Lie derivations on 3-graded associative algebras. We recall that given a unitary commutative ring K, a *3-graded K-algebra* A is a K-algebra which splits into the direct sum $A = A_{-1} \oplus A_0 \oplus A_1$ of nonzero K-submodules satisfying $A_0A_i + A_iA_0 \subset A_i$ for all $i \in \{-1, 0, 1\}$, $A_{-1}A_1 + A_1A_{-1} \subset A_0$ and $A_1A_1 = A_{-1}A_{-1} = 0$. A linear mapping D on a 3-graded associative algebra A such that $D(A_i) \subset A_i$, $i \in \{-1, 0, 1\}$, is called a *graded derivation* if satisfies $D(xy) = D(x)y + xD(y)$ for all $x, y \in A$, and it is called a *graded Lie derivation* if $D([x, y]) = [D(x), y] + [x, D(y)]$ holds for all $x, y \in A$. Here and subsequently, the bracket denotes the Lie product, $[x, y] = xy - yx$ on A. From now on A will denote

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a 3-graded associative algebra defined over a field K of characteristic not 2. We define the *annihilator* of A as the ideal given by $Ann(A) = \{x \in A : xy = yx = 0 \text{ for all } y \in A\}$. Our purpose is to show the following theorem

Theorem 1. *Let D be a graded Lie derivation on a semiprime 3-graded associative algebra $A = A_{-1} \oplus A_0 \oplus A_1$ such that $A_0 = [A_{-1}, A_1]$. Then D is a 3-graded derivation.*

Corollary 1. *Let D be a graded Lie derivation on an infinite dimensional topologically simple 3-graded associative H^*-algebra A. Then D is a 3-graded derivation.*

2. The Theorem

Given a 3-graded associative algebra, we have clearly that (A_{-1}, A_1) is an associative pair with respect to the triple products $< x, y, z >^\sigma = xyz$ for $\sigma = \pm$, with $x, z \in A_{-1}$ and $y \in A_1$ if $\sigma = +$, and with $x, z \in A_1$ and $y \in A_{-1}$ if $\sigma = -$. If $P = (P^+, P^-)$ is an associative pair isomorphic to the associative pair (A_{-1}, A_1), we shall say that A is a 3-graded algebra *envelope* of P if $A_0 = A_{-1}A_1 + A_1A_{-1}$. An envelope A is *tight* if

$$\{x_0 \in A_0 : x_0A_\sigma = A_\sigma x_0 = 0, \sigma = \pm 1\} = 0.$$

A *derivation* D on a, non-necessarily associative, pair $V = (V^+, V^-)$ is a couple of linear mappings $D = (D^+, D^-)$, $D^\sigma : V^\sigma \to V^\sigma$, satisfying

$$D^\sigma(< x^\sigma, y^{-\sigma}, z^\sigma >) = < D^\sigma(x^\sigma), y^{-\sigma}, z^\sigma >$$
$$+ < x^\sigma, D^{-\sigma}(y^{-\sigma}), z^\sigma > + < x^\sigma, y^{-\sigma}, D^\sigma(z^\sigma) >$$

for any $x^\sigma, z^\sigma \in V^\sigma$, $y^{-\sigma} \in V^{-\sigma}$ and $\sigma = \pm$.

Proposition 1. *Let $P = (P^+, P^-)$ be an associative pair, let D be a derivation of P, and let A be a 3-graded tight algebra envelope of P. Then there is a unique graded derivation D' on A extending D.*

Proof. As $A = P^+ \oplus (P^+P^- + P^-P^+) \oplus P^-$, we can define first

$$D' : P^+ \oplus (P^+P^- + P^-P^+) \oplus P^- \to A$$

by writing $D'(x) := D^\sigma(x)$ for all $x \in P^\sigma$, $\sigma = \pm$, and

$$D'(\sum_j (x_j y_j + u_j v_j)) := \sum_j (D^+(x_j) y_j + x_j D^-(y_j) + D^-(u_j) v_j + u_j D^+(v_j))$$

for arbitrary $x_j, v_j \in P^+$ and $y_j, u_j \in P^-$. The definition is correct since if $\sum_j (x_j y_j + u_j v_j) = 0$. Letting

$$z := \sum_j (D^+(x_j) y_j + x_j D^-(y_j) + D^-(u_j) v_j + u_j D^+(v_j)),$$

the equations

$$D^+(\sum_j (x_j y_j + u_j v_j) x) = 0 \quad \text{and} \quad D^-(\sum_j (x_j y_j + u_j v_j) y) = 0$$

for any $x \in P^+$ and any $y \in P^-$ imply $z P^\sigma = 0$, $\sigma = \pm$. In a similar way we have $P^\sigma z = 0$. Hence $z = 0$. The fact that D' is a derivation is easy to check and the proof is complete. □

Proof of Theorem 1. As A is a 3-graded associative algebra, we find that the pair

$$J := ((A_{-1}, A_1), \{\cdot,\cdot,\cdot\}^\sigma)$$

is a Jordan pair in the sense of [10] with respect to the triple products $\{x, y, z\}^\sigma = [[x, y], z]$ for $\sigma = \pm$, with $x, z \in A_{-1}$, $y \in A_1$ if $\sigma = +$ and $x, z \in A_1$, $y \in A_{-1}$ if $\sigma = -$. Hence (D, D) is a derivation of J. It is proved in [11] that any Jordan derivation on a semiprime associative pair over a field K of characteristic not 2 is an associative derivation. As the 3-graduation of A also implies $\{x, y, z\}^\sigma = xyz + zyx$, the above result gives us that (D, D) is a derivation of the associative pair $P := ((A_{-1}, A_1), < \cdot,\cdot,\cdot >^\sigma)$ being $< \cdot,\cdot,\cdot >^\sigma = xyz$. It is easy to check that A is a 3-graded tight algebra envelope of P, hence Proposition 1 shows that (D, D) extends uniquely to a derivation D' on A. We assert that $D = D'$. Indeed, $D(x) = D'(x)$ for any $x \in A_{-1} \cup A_1$ and chosen any $x_0 \in A_0$, the condition $A_0 = [A_{-1}, A_1]$ gives us $D(x_0) = D'(x_0)$. The proof is complete. □

In order to prove Corollary 1, we recall that an *H^*-algebra* A over K, $K = \mathbb{R}$ or $K = \mathbb{C}$, is a non-necessarily associative K-algebra whose underlying vector space is a Hilbert space with inner product

$(\cdot|\cdot)$, endowed either with a linear map if $K=\mathbb{R}$ or with a conjugate-linear map if $K=\mathbb{C}$, $*: A\to A$ $(x\mapsto x^*)$, such that $(x^*)^*=x$, $(xy)^*=y^*x^*$ for any $x,y\in A$ and the following hold

$$(xy|z)=(x|zy^*)=(y|x^*z)$$

for all $x,y,z\in A$. The map $*$ will be called the *involution* of the H^*-algebra. The continuity of the product of A is proved in [8]. We call the H^*-algebra A, *topologically simple* if $A^2\neq 0$ and A has no nontrivial closed ideals. H^*-algebras were introduced and studied by Ambrose [1] in the associative case, and have been also considered in the case of the most familiar classes of nonassociative algebras [4, 6, 8, 9] and even in the general nonassociative context [7, 15]. In [8] it is proved that any H^*-algebra A with continuous involution splits into the orthogonal direct sum $A=Ann(A)\perp\overline{\mathcal{L}(A^2)}$, where $Ann(A)$ denotes the annihilator of A defined as in §1, and $\overline{\mathcal{L}(A^2)}$ is the closure of the vector span of A^2, which turns out to be an H^*-algebra with zero annihilator. Moreover, each H^*-algebra A with zero annihilator satisfies $A=\overline{\perp I_\alpha}$ where $\{I_\alpha\}_\alpha$ denotes the family of minimal closed ideals of A, each of them being a topologically simple H^*-algebra. We also recall that any derivation on arbitrary H^*-algebras with zero annihilator is continuous [15].

Proof of Corollary 1. The structure theories of topologically simple associative and Lie H^*-algebras given in [8] and [6] respectively imply that the antisymmetrized Lie H^*-algebra A^- of A is also a topologically simple Lie H^*-algebra. Hence $A_0=\overline{[A_{-1},A_1]}$. There is not any problem in arguing as in Theorem 1 to prove that $D=D'$ on $A_1\oplus[A_{-1},A_1]\oplus A_1$. By [15], D and D' are continuous and therefore $D=D'$ on $A=A_1\oplus\overline{[A_{-1},A_1]}\oplus A_1$. □

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