BOOK REVIEWS

Operator Spaces

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The motivation of the study of *operator spaces* ties up with the notion of quantization. In fact, this notion started its life with the 'matrix mechanics' of Heisenberg. Influenced by this work of Heisenberg, von Neumann suggested that one should seek 'quantized' analogues of mathematics, in the sense of replacing functions by operators. Murray and von Neumann put this into practise by producing the operator (or quantized) version of integration. This gave birth to the whole field of *operator algebras*. Similarly, one can seek for the notion of 'quantization of Banach spaces' which turns out to be operator spaces. On the other hand, the study of operator spaces is related to the study of complete boundedness of mappings (i.e. the morphisms between operator spaces) which is found to be useful in the study of operator algebras long before the operator space theory was axiomatized.

In fact, operator spaces is an important subject which appears naturally in different areas of Mathematics (especially the study of operator algebras) and has many important applications in those areas. The authors of this book are both experts on the subject and in most topics discussed in the book, they have many substantial and important contributions.

As noted in the above, operator spaces are 'quantized Banach spaces' which basically means that they are spaces of bounded operators on some Hilbert spaces. More precisely, if X is a subspace of

$\mathcal{B}(H)$, then the set of matrices, $M_n(X)$, with coefficients in X can also be regarded as a subspace of $\mathcal{B}(H^n)$. In this way, we obtain a sequence of normed spaces $\{M_n(X), \|\cdot\|_n\}_{n\in\mathbb{N}}$ which satisfies the following conditions: for any $u \in M_m(X)$, $v \in M_n(X)$, $\alpha \in M_{n,m}(\mathbb{C})$ and $\beta \in M_{m,n}(\mathbb{C})$,

(1) $\|u \oplus v\|_{m+n} = \max\{\|u\|_m, \|v\|_n\}$;
(2) $\|\alpha \cdot u \cdot \beta\|_n \leq \|\alpha\| \cdot \|u\|_m \cdot \|\beta\|$.

One of the important contributions of the second author is the abstract characterisation (as given in Chapter 2 of this book) which shows that this sequence of normed spaces together with the compatibility conditions characterise the operator space, i.e. one can find a Hilbert space representation that gives back this sequence (Theorem 2.3.5). This is called the Ruan's representation theorem.

On the other hand, one can group together the sequence $\{M_n(X), \|\cdot\|_n\}_{n\in\mathbb{N}}$ and obtain a norm on the algebraic tensor product $X \odot \mathcal{K}$ (where $\mathcal{K}$ is the space of compact operator on a separable Hilbert space) that satisfies some conditions. Using Ruan's representation theorem, one sees that for a Banach space X, the above norm on $X \odot \mathcal{K}$, together with compatibility conditions, completely characterise the operator space structure on X. In this way, one can also view (complete) operator spaces as a special kind of Banach $\mathcal{K}$-bimodules and this gives another way of seeing operator spaces as quantised Banach spaces (in the sense that the scalar $\mathbb{C}$ is replaced by the "highly non-commutative" algebra $\mathcal{K}$).

There are many applications of the subject in different areas of Mathematics. One of these interesting applications is in harmonic analysis. In particular, there is a natural operator space structure for the Fourier algebra of a locally compact group which proved to be an important structure (e.g. the group is amenable if and only if the Fourier algebra, equipped with such an operator space structure, is "operator amenable"). This is briefly discussed in Chapter 16 of the book.

Apart from the applications in various fields, there are, roughly speaking, two major research directions concerning the study of operator spaces.

The first one is the structure theory. One fruitful approach in this direction is the quantisation philosophy as described above and aims at studying the operator space analogue of some Banach space properties or theories. As operator space is more complicated than

Banach space, this is not an easy task. Chapters 11 – 13, as well as Chapters 14 and 15, of the book illustrate the "quantisation" of some theories of Grothendieck on Banach spaces. A word of warning is that the results in operator spaces and those in Banach spaces are sometimes very different and some Banach space theories do not have direct operator space analogues, e.g. the natural analogue of the principle of local reflexivity does not hold for general operator spaces (but this also makes the subject more interesting).

The second research direction is to look at the interaction between operator space theory and the theory of operator algebras. For example, some properties for operator algebras are defined in the more general framework of operator spaces and operator spaces techniques are used to enhance the original study of the properties. Some of the successful examples are Injectivity, Exactness and Nuclearity. Chapters 6, 14 and 15 give a brief account of these. Moreover, one can use operator space to give an abstract characterisation of non-selfadjoint operator algebras (which is given in Chapter 17).

There are certainly intersections and interactions between the two directions and these make the subject even more interesting. For example, as discussed in Chapters 14 – 15, the principle of local reflexivity and completely integral mappings (which are analogous to the theory of Banach spaces) turn out to be related to exactness. On the other hand, the development of the tensor products of operator spaces were influenced by both the theory of Banach spaces and the theory of operator algebras. In fact, tensor products of operator spaces is one of the most important part of this subject and have many applications to several fields of Mathematics. In this book, three of these important tensor products, namely, the projective, the injective and the Haagerup tensor product are discussed (in Chapters 7 – 10).

Overall, the book is well written and well structured. It assumes very little knowledge on Banach spaces and C^*-algebras and even starts with some basic matrix theory in Chapter one. Chapters 2 and 3 give the basic definitions, constructions and properties of operator spaces (which include the abstract characterisation as mentioned above) as well as some interesting and important examples. In Chapters 4 and 5, the Arveson-Wittstock extension theorem and the Stinespring dilation theorem (as well as some other results) are discussed. From Chapter 6 onward, a number of interesting and important aspects are introduced as mentioned above.

As a final comment, we note that some notation and terminology may not be placed in the natural order (e.g. the notation of $M_{r\times\infty}$ and K_∞ on p.90 are defined in p.176 and p.177, respectively). However, with the good indexes of terminology and notation at the back of the book, the readers will not find it difficult to locate their definitions. I highly recommend this book to anyone who wants to learn about the theory of operator spaces.