C^*-Algebras in Numerical Analysis

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These are the notes for two lectures I gave at the Belfast Functional Analysis Day 1999. The purpose of these notes is to give an idea of how C^*-algebra techniques can be successfully employed in order to solve some concrete problems of Numerical Analysis. I focus my attention on several questions concerning the asymptotic behavior of large Toeplitz matrices. This limitation ignores the potential and the triumphs of C^*-algebra methods in connection with large classes of other operators and plenty of different approximation methods, but it allows me to demonstrate the essence of the C^*-algebra approach and to illustrate it with nevertheless nontrivial examples.

Prologue

The idea of applying C^*-algebras to problems of numerical analysis emerged in the early 1980's. At the beginning of the eighties, Silbermann [54] discovered a new way of translating the problem of the stability of the finite section method for Toeplitz operators into an invertibility problem in Banach algebras. Using powerful Banach algebra techniques, in particular local principles, he was so able to prove a series of spectacular results. Soon it became clear that the prevailing Banach algebras are or can be replaced by C^*-algebras in many interesting situations. As C^*-algebras enjoy a lot of nice properties that are not shared by general Banach algebras, it was possible to sharpen various known results of numerical analysis significantly, to give extremely simple and lucid proofs of several profound theorems, and to open the door to a wealth of new insights and results. The first explicit use of C^*-algebras in connection with a problem of numerical analysis was probably made in the paper [17] by Silbermann and myself.

Meanwhile the application of C^*-algebras to numerical analysis has grown to a big business. Here, I confine myself with quoting Roch and Silbermann's paper [51], Arveson's articles [3], [4], and Hagen, Roch, and Silbermann's monographs "Spectral Theory of Approximation Methods for Convolution Operators" and "C^*-Algebras and Numerical Analysis" ([33] and [34]).

In connection with the numerical analysis of Toeplitz matrices, C^*-algebras are substantially used in my books [18] and [19] with Silbermann. In a sense, the present text is an extract of some ideas of these books, supplemented and completed by some recent ideas of Roch and Silbermann.

I am very grateful to Roland Hagen, Steffen Roch, and Bernd Silbermann for providing me with the manuscript of their book [34] and allowing me to benefit from this inexhaustible source when preparing these notes. I am also greatly indebted to Martin Mathieu and Anthony W. Wickstead for their perfect organization of the Belfast Functional Analysis Day 1999 and for inviting me to write these notes. The first Belfast Functional Analysis Day took place in 1998, and I would be happy if the Belfast Functional Analysis Day would become a traditional annual meeting throughout the years to come, with the same pleasant and stimulating atmosphere as this time.

1. Finite sections of infinite matrices

Let $\mathcal{B}(l^2)$ denote the set of all bounded linear operators on the Hilbert space $l^2 := l^2(\{1, 2, 3, \ldots\})$. Given $A \in \mathcal{B}(l^2)$, we consider the equation

$$Ax = y. \tag{1}$$

This equation amounts to a linear system with infinitely many equations and an infinite number of unknowns:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \ldots \\ a_{21} & a_{22} & a_{23} & \ldots \\ a_{31} & a_{32} & a_{33} & \ldots \\ \vdots & \vdots & \vdots & \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \end{pmatrix}. \tag{2}$$

We replace the infinite system (2) by the finite system

$$\begin{pmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \ldots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1^{(n)} \\ \vdots \\ x_n^{(n)} \end{pmatrix} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}. \tag{3}$$

Passage from (2) to (3) is a special projection method. For $n = 1, 2, 3, \ldots$ define the projection P_n by

$$P_n : l^2 \to l^2,\ (x_1, x_2, x_3, \ldots) \mapsto (x_1, x_2, \ldots, x_n, 0, 0, \ldots). \tag{4}$$

In what follows we will freely identify $\operatorname{Im} P_n$, the image of P_n, with $\mathbf{C}^n$. In particular, we always think of $\mathbf{C}^n$ as being equipped with the l^2 norm. The matrix

$$A_n := \begin{pmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \ldots & a_{nn} \end{pmatrix} \tag{5}$$

can now be identified with $P_n A P_n$, and equation (3) can be written in the form

$$A_n x^{(n)} = P_n y, \quad x^{(n)} \in \mathbf{C}^n. \tag{6}$$

Convergence of the finite section method. Suppose the operator A is invertible. Are the matrices A_n invertible for all sufficiently large n and do, for every $y \in l^2$, the solutions $x^{(n)}$ of (6) converge to the solution x of (1) ? Here we regard $x^{(n)}$ as an element of l^2, and convergence of $x^{(n)}$ to x means that $x^{(n)} \to x$ in l^2. If the answer to the above question is yes, then one says that the *finite section method is convergent for the operator A.* Equivalently, the finite section method converges if and only if the matrices A_n are invertible for all sufficiently large n and if A_n^{-1} converges strongly (= pointwise) to A^{-1}. In this and similar contexts, A_n^{-1} is thought of as being extended by zero from $\mathbf{C}^n$ to all of l^2, so that A_n^{-1} may be considered as an operator on l^2. Thus, the strong convergence $A_n^{-1} \to A^{-1}$ actually means that $A_n^{-1} P_n y \to A^{-1} y$ for all $y \in l^2$.

Stability. A sequence $\{A_n\}_{n=1}^{\infty}$ of $n \times n$ matrices A_n is said to be *stable* if the matrices A_n are invertible for all sufficiently large n, say $n \geq n_0$, and if

$$\sup_{n \geq n_0} \|A_n^{-1}\| < \infty.$$

Here $\|\cdot\|$ is the operator norm on $\mathbf{C}^n$ associated with the l^2 norm (in other terms: $\|\cdot\|$ is the spectral norm). Throughout what follows we put $\|B^{-1}\| = \infty$ if B is not invertible. With this convention, we can say that the sequence $\{A_n\}_{n=1}^{\infty}$ is stable if and only if

$$\limsup_{n \to \infty} \|A_n^{-1}\| < \infty.$$

The following fact is well known and easily proved.

Proposition 1.1. *If A is invertible, then the finite section method for A is convergent if and only if the sequence $\{A_n\}_{n=1}^{\infty}$ consisting of the matrices* (5) *is stable.* ■

Notice that this result is a concretisation of the general numerical principle

$$\text{convergence} \;=\; \text{approximation} \;+\; \text{stability}.$$

Since $A_n = P_n A P_n \to A$ strongly, the approximation property is automatically satisfied, and hence the question whether the finite section method converges comes completely down to the question whether the sequence $\{A_n\}_{n=1}^{\infty}$ is stable.

As the following result shows, the sequence of the matrices (5) is never stable if A is not invertible.

Proposition 1.2. *If the sequence $\{A_n\}_{n=1}^{\infty}$ of the matrices* (5) *is stable, then A is necessarily invertible.*

Proof. Let $\|A_n^{-1}\| \le M$ for $n \ge n_0$. Then if $x \in l^2$ and $n \ge n_0$,

$$\|P_n x\| = \|A_n^{-1} A_n x\| \le M \|A_n x\| = M \|P_n A P_n x\|,$$
$$\|P_n x\| = \|(A_n^*)^{-1} A_n^* x\| \le M \|A_n^* x\| = M \|P_n A^* P_n x\|,$$

and passing to the limit $n \to \infty$, we get

$$\|x\| \le M\|Ax\|, \quad \|x\| \le M\|A^* x\| \tag{7}$$

for every $x \in l^2$. This shows that A is invertible. ■

Spectral approximation. The spectrum $\operatorname{sp} B$ of a bounded linear operator B is defined as usual:

$$\operatorname{sp} B := \{\lambda \in \mathbf{C} : B - \lambda I \ \text{ is not invertible}\}.$$

For $A \in \mathcal{B}(l^2)$, let the matrices A_n be given by (5). What is the relation between the spectra (sets of eigenvalues) of the matrices A_n and the spectrum of the operator A? Do the eigenvalues of A_n for large n, for $n = 1000$ say, tell us anything about the spectrum of A?

Or conversely, if the spectrum of A is known, does this provide any piece of information about the eigenvalues of A_n for very large n?

Condition numbers. Again let $A \in \mathcal{B}(l^2)$ and define A_n by (5). What can be said about the connection between the condition number

$$\kappa(A) := \|A\| \, \|A^{-1}\|$$

and the condition numbers

$$\kappa(A_n) := \|A_n\| \, \|A_n^{-1}\|$$

for large n ? Clearly, this question is much more "numerical" than the question about the sole stability of the sequence $\{A_n\}_{n=1}^{\infty}$. Since $\|A_n\| = \|P_n A P_n\| \to \|A\|$ as $n \to \infty$, the question considered here amounts to the question whether $\|A_n^{-1}\|$ is close to $\|A^{-1}\|$ for sufficiently large n.

Proposition 1.3. *If the sequence $\{A_n\}_{n=1}^{\infty}$ of the matrices* (5) *is stable, then*

$$\|A^{-1}\| \le \liminf_{n\to\infty} \|A_n^{-1}\|. \tag{8}$$

Proof. If $\|A_{n_k}^{-1}\| \le M$ for infinitely many n_k, the argument of the proof of Proposition 1.2 yields (7) and thus (8). ∎

2. Compact operators

The answers to the questions raised in the preceding section are well known in the case where $A = I + K$ with some compact operator K. Let $\mathcal{K}(l^2)$ denote the collection of all compact operators on l^2. In what follows, P_n always stands for the projection defined by (4). Notice that P_n is the identity operator on $\mathbf{C}^n$; to emphasise this fact, we write $I + P_n K P_n$ for $P_n + P_n K P_n$.

Proposition 2.1. *Let $K \in \mathcal{K}(l^2)$. The sequence $\{I + P_n K P_n\}_{n=1}^{\infty}$ is stable if and only if $I + K$ is invertible. Moreover, we have*

$$\lim_{n\to\infty} \|(I + P_n K P_n)^{-1}\| = \|(I + K)^{-1}\|.$$

This follows easily from the observation that the compactness of K implies that $P_n K P_n$ converges uniformly (i.e., in the norm topology) to K. ∎

Things are a little bit more intricate when considering the problem of spectral approximation. We first recall two standard definitions. Given a sequence $\{E_n\}_{n=1}^{\infty}$ of sets $E_n \subset \mathbf{C}$, the *uniform limiting set*

$$\liminf_{n\to\infty} E_n$$

is defined as the set of all $\lambda \in \mathbf{C}$ for which there are $\lambda_1 \in E_1, \lambda_2 \in E_2, \lambda_3 \in E_3, \dots$ such that $\lambda_n \to \lambda$, while the *partial limiting set*

$$\limsup_{n\to\infty} E_n$$

is the set of all $\lambda \in \mathbf{C}$ for which there exist $n_1 < n_2 < n_3 < \dots$ and $\lambda_{n_k} \in E_{n_k}$ such that $\lambda_{n_k} \to \lambda$. Obviously,

$$\liminf_{n\to\infty} E_n \subset \limsup_{n\to\infty} E_n.$$

If operators $A_n \in \mathcal{B}(l^2)$ converge uniformly to some operator $A \in \mathcal{B}(l^2)$, then, by the upper semi-continuity of the spectrum,

$$\limsup_{n\to\infty} \operatorname{sp} A_n \subset \operatorname{sp} A, \tag{9}$$

but in general equality need not hold in (9). This phenomenon does not occur for compact operators.

Proposition 2.2. *If $K \in \mathcal{K}(l^2)$, then*

$$\liminf_{n\to\infty} \operatorname{sp}(I + P_nKP_n) = \limsup_{n\to\infty} \operatorname{sp}(I + P_nKP_n) = \operatorname{sp}(I + K).$$

Proof (*after Torsten Ehrhardt*). A moment's thought reveals that it suffices to show that

$$\liminf_{n\to\infty} \operatorname{sp}(P_nKP_n) \supset \operatorname{sp} K.$$

As K is compact, $\operatorname{sp} K$ is an at most countable set and the origin is the only possible accumulation point of $\operatorname{sp} K$. Fix an isolated point λ_0 in $\operatorname{sp} K$. If $\varepsilon > 0$ is sufficiently small, then

$$\|(P_nKP_n - \lambda I)^{-1} - (K - \lambda I)^{-1}\| \to 0 \ \text{ as } \ n \to \infty$$

uniformly with respect to all λ on the circle $|\lambda - \lambda_0| = \varepsilon$. Hence, the Riesz projections

$$\Pi_n := \frac{1}{2\pi i} \int_{|\lambda-\lambda_0|=\varepsilon} (P_nKP_n - \lambda I)^{-1}\, d\lambda$$

converge in the norm to the Riesz projection

$$\Pi := \frac{1}{2\pi i} \int_{|\lambda-\lambda_0|=\varepsilon} (K - \lambda I)^{-1}\, d\lambda.$$

The projection Π is nonzero, because λ_0 is in $\operatorname{sp} K$. It follows that the projections Π_n are nonzero for all sufficiently large n, which implies that the disk $|\lambda - \lambda_0| < \varepsilon$ contains a point of $\operatorname{sp}(P_n K P_n)$ for every n large enough. As $\varepsilon > 0$ can be chosen arbitrarily small, it results that

$$\lambda_0 \in \liminf_{n\to\infty} \operatorname{sp}(P_n K P_n).$$

This completes the proof in the case where the origin is an isolated point of $\operatorname{sp} K$. If the origin is an accumulation point of $\operatorname{sp} K$, it must belong to $\liminf \operatorname{sp}(P_n K P_n)$ because this set is closed. ■

3. Selfadjoint operators

In the case of bounded selfadjoint operators, partial answers to the questions raised in Section 1 can be given by invoking standard functional analysis. Throughout this section we assume that $A \in \mathcal{B}(l^2)$ is a selfadjoint operator, $A = A^*$, and that $A_n = P_n A P_n$ is defined by (5).

Proposition 3.1. *There exist selfadjoint and invertible operators $A \in \mathcal{B}(l^2)$ for which the sequence $\{A_n\}_{n=1}^{\infty}$ is not stable. Moreover, given any numbers a, b such that $0 < a \le b \le \infty$, there exists a selfadjoint operator $A \in \mathcal{B}(l^2)$ such that*

$$\|A^{-1}\| = a, \quad \limsup_{n\to\infty} \|A_n^{-1}\| = b.$$

(In connection with the requirement $a \le b$, recall Proposition 1.3.)

Proof. Put

$$B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } A = \operatorname{diag}(B, B, B, \ldots).$$

Then A is selfadjoint and invertible, but the matrices A_n contain a zero column and are therefore not invertible whenever n is odd. This shows that the sequence $\{A_n\}_{n=1}^{\infty}$ is not stable.

Now let
$$C = \begin{pmatrix} 1/b & 1/a+1/b \\ 1/a+1/b & 1/b \end{pmatrix}$$
and
$$A = \mathrm{diag}\,(C, C, C, \ldots).$$
The eigenvalues of C are $\lambda_1 = 1/a$ and $\lambda_2 = 1/a + 2/b$, whence
$$\|A^{-1}\| = \|C^{-1}\| = 1/\lambda_1 = a,$$
and
$$\|A_n^{-1}\| = \begin{cases} \max(\|C^{-1}\|, b) = \max(a, b) = b & \text{if} \quad n \ \text{ is odd}, \\ \|C^{-1}\| = a & \text{if} \quad n \ \text{ is even}. \end{cases}$$
Thus, $\limsup \|A_n^{-1}\| = b$. ■

As for stability and convergence of the condition numbers, everything is well for definite operators.

Proposition 3.2. *Let $A = A^* \in \mathcal{B}(l^2)$ be positive definite,*
$$(Ax, x) \geq \varepsilon \|x\|^2$$
with some $\varepsilon > 0$ for all $x \in l^2$. Then $\{A_n\}_{n=1}^\infty$ is stable and we have
$$\|A_n\| \leq \|A\| \ \textit{ for all } \ n \geq 1, \qquad \lim_{n\to\infty} \|A_n\| = \|A\|,$$
$$\|A_n^{-1}\| \leq \|A^{-1}\| \ \textit{ for all } \ n \geq 1, \qquad \lim_{n\to\infty} \|A_n^{-1}\| = \|A^{-1}\|.$$

Proof. Put
$$m = \inf_{x \neq 0} \frac{(Ax, x)}{(x, x)}, \quad M = \sup_{x \neq 0} \frac{(Ax, x)}{(x, x)}, \tag{10}$$
$$m_n = \inf_{x \neq 0} \frac{(A_n P_n x, P_n x)}{(P_n x, P_n x)}, \quad M_n = \sup_{x \neq 0} \frac{(A_n P_n x, P_n x)}{(P_n x, P_n x)}. \tag{11}$$
By assumption, $m \geq \varepsilon > 0$. We have
$$m_n = \inf_{x \neq 0} \frac{(P_n A P_n x, P_n x)}{(P_n x, P_n x)} = \inf_{x \neq 0} \frac{(A P_n x, P_n x)}{(P_n x, P_n x)} \geq \inf_{y \neq 0} \frac{(Ay, y)}{(y, y)} = m,$$
and, analogously, $M_n \leq M$. Because
$$\|A\| = M, \quad \|A_n\| = M_n, \quad \|A^{-1}\| = 1/m, \quad \|A_n^{-1}\| = 1/m_n,$$

we arrive at the inequalities $\|A_n\| \le \|A\|$ and $\|A_n^{-1}\| \le \|A^{-1}\|$ for all n. It is clear that $\|A_n\| \to \|A\|$, and since

$$\|A^{-1}\| \le \liminf_{n\to\infty} \|A_n^{-1}\| \le \limsup_{n\to\infty} \|A_n^{-1}\| \le \|A^{-1}\|,$$

it follows that $\|A_n^{-1}\| \to \|A^{-1}\|$. ■

Spectral approximation is a delicate problem for general selfadjoint operators, and the reader is referred to [3], [4], [34] for this topic. We here confine ourselves to a few simple remarks.

Given a selfadjoint operator $A \in \mathcal{B}(l^2)$, define the real numbers m and M by (10). Furthermore, let $\lambda_{\min}(A_n)$ and $\lambda_{\max}(A_n)$ be the minimal and maximal eigenvalues of A_n, respectively. Of course, with m_n and M_n given by (11), we have

$$\lambda_{\min}(A_n) = m_n, \quad \lambda_{\max}(A_n) = M_n.$$

Proposition 3.3. *If $A \in \mathcal{B}(l^2)$ is selfadjoint, then*

$$m \le \lambda_{\min}(A_n) \le \lambda_{\max}(A_n) \le M, \tag{12}$$

$$\lim_{n\to\infty} \lambda_{\min}(A_n) = m, \quad \lim_{n\to\infty} \lambda_{\max}(A_n) = M, \tag{13}$$

$$\{m, M\} \subset \operatorname{sp} A \subset \liminf_{n\to\infty} \operatorname{sp} A_n \subset \limsup_{n\to\infty} \operatorname{sp} A_n \subset [m, M]. \tag{14}$$

Proof. The validity of (12) and (13) was established in the proof of Proposition 3.2. The only nontrivial part of (14) is the inclusion

$$\operatorname{sp} A \subset \liminf_{n\to\infty} \operatorname{sp} A_n. \tag{15}$$

So assume $\lambda \in \mathbf{R}$ is not in $\liminf \operatorname{sp} A_n$. Then there is an $\varepsilon > 0$ such that

$$U_\varepsilon(\lambda) \cap \operatorname{sp} A_n = \emptyset \ \text{ for all } \ n \ge n_0,$$

where $U_\varepsilon(\lambda) := \{z \in \mathbf{C} : |z - \lambda| < \varepsilon\}$. Hence $U_\varepsilon(0) \cap \operatorname{sp}(A_n - \lambda I) = \emptyset$ for all $n \ge n_0$, and since $(A_n - \lambda I)^{-1}$ is Hermitian, and therefore its norm coincides with the spectral radius, we get

$$\|(A_n - \lambda I)^{-1}\| \le 1/\varepsilon \ \text{ for all } \ n \ge n_0.$$

It follows that $\{A_n - \lambda I)_{n \ge n_0}$ is stable, and therefore $A - \lambda I$ must be invertible (Proposition 1.2). Consequently, $\lambda \notin \operatorname{sp} A$, which completes the proof of (15). ■

Example 3.4 (trivial). The limiting set limsup A_n may be much smaller than $[m,M]$: if $A = \operatorname{diag}(0,1,1,1,\ldots)$, then $m=0$, $M=1$, $\operatorname{sp} A = \{0,1\}$, and $\operatorname{sp} A_n = \{0,1\}$ for all $n \geq 2$. ■

Example 3.5 (less trivial). There are selfadjoint operators $A \in \mathcal{B}(l^2)$ for which the limiting set $\liminf \operatorname{sp} A_n$ is much larger than $\operatorname{sp} A$. To construct an example, let a_n be the nth Fourier coefficient of the characteristic function of the upper half of the complex unit circle $\mathbf{T}$,

$$a_n = \frac{1}{2\pi}\int_0^\pi e^{-in\theta}d\theta = \frac{1-(-1)^n}{2\pi i n} = \begin{cases} 1/2 & \text{if} \quad n=0 \\ 1/(\pi i n) & \text{if} \quad n \text{ is odd,} \\ 0 & \text{if} \quad n \neq 0 \text{ is even,} \end{cases}$$

and put

$$A = \left(\begin{array}{c|cc|cc|cc|c} a_0 & a_{-1} & a_1 & a_{-2} & a_2 & a_{-3} & a_3 & \ldots \\ \hline a_1 & a_0 & a_2 & a_{-1} & a_3 & a_{-2} & a_4 & \ldots \\ a_{-1} & a_{-2} & a_0 & a_{-3} & a_1 & a_{-4} & a_2 & \ldots \\ \hline a_2 & a_1 & a_3 & a_0 & a_4 & a_{-1} & a_5 & \ldots \\ a_{-2} & a_{-3} & a_{-1} & a_{-4} & a_0 & a_{-5} & a_1 & \ldots \\ \hline a_3 & a_2 & a_4 & a_1 & a_5 & a_0 & a_6 & \ldots \\ a_{-3} & a_{-4} & a_{-2} & a_{-5} & a_{-1} & a_{-6} & a_0 & \ldots \\ \hline \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{array}\right).$$

One can show that A induces a bounded selfadjoint operator on l^2 for which

$$\operatorname{sp} A = \{0,1\},$$
$$\liminf_{n\to\infty} \operatorname{sp} A_n = \limsup_{n\to\infty} \operatorname{sp} A_n = [0,1].$$

We will return to this example (and give the mystery's resolution) in Section 12. ■

4. Toeplitz operators

These are the operators on l^2 generated by matrices of the form

$$(a_{j-k})_{j,k=1}^{\infty} = \begin{pmatrix} a_0 & a_{-1} & a_{-2} & \dots \\ a_1 & a_0 & a_{-1} & \dots \\ a_2 & a_1 & a_0 & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}, \tag{16}$$

that is, by matrices which have equal entries on the parallels to the main diagonal. Notice that (16) is completely specified by the complex numbers $\{a_n\}_{n=-\infty}^{\infty}$ in the first row and the first column.

Theorem 4.1 (Toeplitz 1911). *The matrix* (16) *induces a bounded operator on l^2 if and only if the numbers a_n $(n \in \mathbf{Z})$ are the Fourier coefficients of some essentially bounded function, i.e., if and only if there exists a function a in L^∞ on the complex unit circle $\mathbf{T}$ such that*

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} a(e^{i\theta})e^{-in\theta}\, d\theta \ \text{ for all } \ n \in \mathbf{Z}.$$

A proof can be found in [18, Theorem 2.7], for example.

If the function $a \in L^\infty := L^\infty(\mathbf{T})$ of the preceding theorem exists, it is unique (as an equivalence class of L^∞). This function is usually referred to the *symbol* of the operator (or of the matrix) (16), and in what follows we denote this operator/matrix by $T(a)$.

In the case where a is a trigonometric polynomial,

$$a(e^{i\theta}) = \sum_{k=-N}^{N} a_k e^{ik\theta},$$

the matrix $T(a)$ is a band matrix. Rational functions a (without poles on $\mathbf{T}$) induce Toeplitz matrices whose entries decay exponentially: $a_n = O(e^{-\delta|n|})$ with some $\delta > 0$. In these two cases we have to deal with continuous symbols. The symbol of the so-called Cauchy-Toeplitz matrix

$$\left(\frac{1}{j-k+\gamma}\right)_{j,k=1}^{\infty} = \begin{pmatrix} \frac{1}{\gamma} & \frac{1}{-1+\gamma} & \frac{1}{-2+\gamma} & \dots \\ \frac{1}{1+\gamma} & \frac{1}{\gamma} & \frac{1}{-1+\gamma} & \dots \\ \frac{1}{2+\gamma} & \frac{1}{1+\gamma} & \frac{1}{\gamma} & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix} \quad (\gamma \in \mathbf{C} \setminus \mathbf{Z})$$

is the function

$$a_\gamma(e^{i\theta}) = \frac{\pi}{\sin \pi\gamma} e^{i\pi\gamma} e^{-i\gamma\theta}.$$

This is a piecewise continuous function with a single jump. The jump is at $e^{i\theta} = 1$:

$$a_\gamma(1+0) = \frac{\pi}{\sin \pi\gamma} e^{i\pi\gamma}, \quad a_\gamma(1-0) = \frac{\pi}{\sin \pi\gamma} e^{-i\pi\gamma}.$$

Proposition 4.2. *Let $a \in L^\infty$.*

(a) *The operator $T(a)$ is compact if and only if a vanishes identically.*

(b) *The operator $T(a)$ is selfadjoint if and only if a is real-valued.*

Proof. Part (b) is trivial. To prove part (a), consider the projections $Q_n = I - P_n$:

$$Q_n : l^2 \to l^2, \ (x_1, x_2, x_3, \dots) \mapsto (0, \dots, 0, x_{n+1}, x_{n+2}, \dots).$$

Obviously, $Q_n \to 0$ strongly and therefore, if $T(a)$ is compact, $\|Q_n T(a) Q_n\| \to 0$. But $Q_n T(a) Q_n | \operatorname{Im} Q_n$ results from $T(a)$ by erasing the first n rows and columns and hence $Q_n T(a) Q_n | \operatorname{Im} Q_n$ has the same matrix as $T(a)$. Consequently,

$$\|T(a)\| = \|Q_n T(a) Q_n | \operatorname{Im} Q_n\| = \|Q_n T(a) Q_n\| = o(1),$$

implying that $T(a)$ is the zero operator. ■

Proposition 4.2 tells us that the standard arguments we employed in Sections 2 and 3 for compact and selfadjoint operators are not applicable to Toeplitz operators with properly complex-valued symbols. In fact Toeplitz operators are a very beautiful source for illustrating and motivating several advanced topics of functional analysis and operator theory, including the index theory of Fredholm operators and the use of C^*-algebras.

An operator $A \in \mathcal{B}(l^2)$ is said to be *Fredholm* if it is invertible modulo compact operators, that is, if the coset $A + \mathcal{K}(l^2)$ is invertible in the Calkin algebra $\mathcal{B}(l^2)/\mathcal{K}(l^2)$. The *essential spectrum* $\operatorname{sp}_{\mathrm{ess}} A$ is defined as the set

$$\operatorname{sp}_{\mathrm{ess}} A := \{\lambda \in \mathbf{C} : A - \lambda I \ \text{ is not Fredholm}\}.$$

One can show that $A \in \mathcal{B}(l^2)$ is Fredholm if and only if the kernel

$$\operatorname{Ker} A := \{x \in l^2 : Ax = 0\}$$

is of finite dimension and the image (= range)

$$\operatorname{Im} A := \{Ax : x \in l^2\}$$

is a closed subspace of l^2 with a finite-dimensional complement in l^2. If $A \in \mathcal{B}(l^2)$ is Fredholm, the integer

$$\operatorname{Ind} A := \dim \operatorname{Ker} A - \dim (l^2/\operatorname{Im} A)$$

is referred to as the *index* of A.

Theorem 4.3 (Coburn 1966). *Let $a \in L^\infty$. The operator $T(a)$ is invertible if and only if it is Fredholm of index zero.*

Proofs of this theorem can be found in [18, Theorem 2.38] or [19, Theorem 1.10], for example.

Theorem 4.3 splits the problem to study invertibility of Toeplitz operators into two "simpler tasks": into finding Fredholm criteria and into establishing index formulas.

Continuous symbols. Let $C := C(\mathbf{T})$ be the set of all (complex-valued) continuous function on $\mathbf{T}$. We always think of $\mathbf{T}$ as being oriented in the counter-clockwise sense. If $a \in C$ has no zeros on $\mathbf{T}$, we denote by $\operatorname{wind} a$ the winding number of the closed, continuous and naturally oriented curve $a(\mathbf{T})$ with respect to the origin.

Theorem 4.4 (Gohberg 1952). *Let $a \in C$. The operator $T(a)$ is Fredholm if and only if a has no zeros on $\mathbf{T}$. In that case*

$$\operatorname{Ind} T(a) = -\operatorname{wind} a.$$

Proofs can be found in [18, Theorem 2.42], [19, Theorem 1.17], and many other works devoted to Toeplitz and related operators. Since $T(a) - \lambda I = T(a - \lambda)$, Theorems 4.3 and 4.4 imply that if $a \in C$, then

$$\begin{aligned} \operatorname{sp_{ess}} T(a) &= a(\mathbf{T}), \\ \operatorname{sp} T(a) &= a(\mathbf{T}) \cup \Big\{\lambda \in \mathbf{C} \setminus a(\mathbf{T}) : \operatorname{wind}(a - \lambda) \neq 0\Big\}. \end{aligned}$$

Piecewise continuous symbols. We denote by $PC := PC(\mathbf{T})$ the set of all (complex-valued) piecewise continuous functions on $\mathbf{T}$. Thus, $a \in PC$ if and only if $a \in L^\infty$ and the one-sided limits

$$a(t \pm 0) := \lim_{\varepsilon \to 0 \pm 0} a(te^{i\varepsilon})$$

exist at every point $t = e^{i\theta} \in \mathbf{T}$. We remark that PC is a closed subset (even a closed subalgebra) of L^∞. If $a \in PC$, then for each $\varepsilon > 0$ the set

$$\Big\{t \in \mathbf{T} : |a(t+0) - a(t-0)| > \varepsilon\Big\}$$

is finite. In particular, functions in PC have at most countably many jumps.

The essential range $\mathcal{R}(a)$ of a function $a \in L^\infty$ is the spectrum of a as an element of the Banach algebra L^∞. If $a \in PC$, then the essential range is $\mathcal{R}(a) = \{a(t \pm 0) : t \in \mathbf{T}\}$. Given $a \in PC$, we denote by $a^{\#}(\mathbf{T})$ the closed continuous and naturally oriented curve that results from $\mathcal{R}(a)$ by filling in the line segment $[a(t-0), a(t+0)]$ whenever $a(t-0) \neq a(t+0)$. If $a^{\#}(\mathbf{T})$ does not contain the origin, we let $\operatorname{wind} a^{\#}$ stand for the winding number of the curve $a^{\#}(\mathbf{T})$ about the origin.

Theorem 4.5. *Let $a \in PC$. The operator $T(a)$ is Fredholm if and only if $0 \notin a^{\#}(\mathbf{T})$. In that case*

$$\operatorname{Ind} T(a) = -\operatorname{wind} a^{\#}.$$

This result was discovered by many people, including Calderón, Spitzer, Widom, Gohberg, Krupnik, and Simonenko. Full proofs are in [18, Theorem 2.74] or [19, Theorem 1.23], for instance. In the language of spectra, Theorems 4.3 and 4.5 show that if $a \in PC$, then

$$\begin{aligned} \operatorname{sp}_{\mathrm{ess}} T(a) &= a^{\#}(\mathbf{T}), \\ \operatorname{sp} T(a) &= a^{\#}(\mathbf{T}) \cup \Big\{\lambda \in \mathbf{C} \setminus a^{\#}(\mathbf{T}) : \operatorname{wind}(a-\lambda)^{\#} \neq 0\Big\}. \end{aligned}$$

Large finite Toeplitz matrices. The above theorems tell us a lot about Toeplitz operators, that is, about infinite Toeplitz matrices.

For a natural number, consider the $n \times n$ Toeplitz matrix

$$T_n(a) := (a_{j-k})_{j,k=1}^n = \begin{pmatrix} a_0 & a_{-1} & \dots & a_{-(n-1)} \\ a_1 & a_0 & \dots & a_{-(n-2)} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n-1} & a_{n-2} & \dots & a_0 \end{pmatrix}.$$

Our goal is to study the problems formulated in Section 1 for Toeplitz operators. Thus, we want criteria for the stability of the sequence $\{T_n(a)\}_{n=1}^\infty$ and we are interested in the behavior of $\operatorname{sp} T_n(a)$ and $\kappa(T_n(a))$ for large n. The purpose of what follows is to demonstrate how these questions can be tackled with the help of C^*-algebras.

5. *C**-algebras

In this section we summarize a few results on C^*-algebras that will be needed in the following. Most of the results are cited without a source because they are well known and can be found in the standard books. My favorite references are Arveson [2], Dixmier [24], Douglas [25], Fillmore [27], Mathieu [42], and Murphy [44].

C^*-algebras

A *Banach algebra* is a complex Banach space A with an associative and distributive multiplication satisfying $\|ab\| \le \|a\|\,\|b\|$ for all $a, b \in A$. If a Banach algebra has a unit element, which will be denoted by $1, e$ or I, then it is referred to as a *unital Banach algebra*. A conjugate-linear map $a \mapsto a^*$ of a Banach algebra A into itself is called an *involution* if $a^{**} = a$ and $(ab)^* = b^*a^*$ for all $a, b \in A$. Finally, a C^*-*algebra* is a Banach algebra with an involution subject to the condition $\|a\|^2 = \|a^*a\|$ for all $a \in A$. C^*-algebras are especially nice Banach algebras, and they have a lot of properties that are not owned by general Banach algebras.

Examples

We will see many examples of C^*-algebras in the forthcoming sections. We here only remark that if H is a Hilbert space, then $\mathcal{B}(H)$, the set of all bounded linear operators on H, and $\mathcal{K}(H)$, the collection of all compact linear operators on H, are C^*-algebras under the operator norm and with passage to the adjoint as involution. The sets L^∞, C, PC are C^*-algebras under the $\|\cdot\|_\infty$ norm and the involution $a \mapsto \overline{a}$ (passage to the complex conjugate). The C^*-algebras L^∞, C, PC are commutative, the C^*-algebras $\mathcal{B}(H)$ and $\mathcal{K}(H)$ are not commutative in case $\dim H \geq 2$.

Spectrum of an element

The *spectrum* of an element a of a unital Banach algebra A with the unit element e is the compact and nonempty set

$$\operatorname{sp}_A a := \Big\{\lambda \in \mathbf{C} : a - \lambda e \text{ is not invertible in } A\Big\}.$$

Of course, invertibility of an element $b \in A$ means the existence of an element $c \in A$ such that $bc = cb = e$.

C^*-subalgebras

A subset B of a C^*-algebra A is called a C^*-*subalgebra* if B itself is a C^*-algebra with the norm and the operations of A. The following useful result tells us that C^*-algebras are *inverse closed*.

Proposition 5.1. *If A is a unital C^*-algebra and B is a C^*-subalgebra of A which contains the unit of A, then $\operatorname{sp}_B b = \operatorname{sp}_A b$ for every $b \in B$.*

By virtue of this proposition, we will abbreviate $\operatorname{sp}_A a$ to $\operatorname{sp} a$.

Ideals

A C^*-subalgebra I of a C^*-algebra A is called a *closed ideal* of A if $aj \in I$ and $ja \in I$ for all $a \in A$ and all $j \in I$.

Proposition 5.2. *If I_1 and I_2 are closed ideals of a C^*-algebra A, then their sum*

$$I_1 + I_2 := \{j_1 + j_2 : j_1 \in I_1,\ j_2 \in I_2\}$$

is also a closed ideal of A.

Quotient algebras

If A is a C^*-algebra and I is a closed ideal of A, then the *quotient algebra* A/I is a C^*-algebra with the usual operations and the norm

$$\|a + I\| := \inf_{j \in I} \|a + j\|.$$

Morphisms

A $*$-*homomorphism* is a linear map $\varphi : A \to B$ of a C^*-algebra A into a C^*-algebra B satisfying $\varphi(a)^* = \varphi(a^*)$ and $\varphi(ab) = \varphi(a)\varphi(b)$ for all $a, b \in A$. In case A and B are unital, we also require that a $*$-homomorphism maps the unit of A to the unit of B. Bijective $*$-homomorphisms are referred to as $*$-*isomorphisms*.

Proposition 5.3. *Let A and B be C^*-algebras and let $\varphi : A \to B$ be a $*$-homomorphism. Then the following hold.*

(a) *The map φ is contractive: $\|\varphi(a)\| \le \|a\|$ for all $a \in A$.*

(b) *The image $\varphi(A)$ is a C^*-subalgebra of B.*

(c) *If φ is injective, then φ is an isometry: $\|\varphi(a)\| = \|a\|$ for all $a \in A$.*

(d) *If A and B are unital and φ is injective, then φ preserves spectra:* $\operatorname{sp} \varphi(a) = \operatorname{sp} a$ *for all $a \in A$.*

(e) *If A and B are unital and φ preserves spectra, then φ also preserves norms: $\|\varphi(a)\| = \|a\|$ for all $a \in A$.*

Gelfand theory

Let A be a commutative unital C^*-algebra. A closed ideal I of A is called a *maximal ideal* if $I \ne A$ and if A and I themselves are the only closed ideals of A which contain the set I. A *multiplicative linear functional* is a $*$-homomorphism of A into the C^*-algebra $\mathbf{C}$ of all complex numbers. The map

$$\varphi \mapsto \operatorname{Ker} \varphi := \{a \in A : \varphi(a) = 0\}$$

is a bijection between the set of the multiplicative linear functionals of A and the set of the maximal ideals of A, and therefore maximal ideals and multiplicative linear functionals are freely identified. Let $M(A)$ stand for the set of all multiplicative linear functionals of A. The coarsest topology on $M(A)$ for which the maps

$$M(A) \to \mathbf{C}, \quad \varphi \mapsto \varphi(a)$$

are continuous for all $a \in A$ is referred to as the Gelfand topology, and $M(A)$ equipped with the Gelfand topology is called the *maximal ideal space* of A. The map

$$\Gamma : A \to C(M(A)), \quad (\Gamma a)(\varphi) = \varphi(a)$$

is known as the *Gelfand transform* of A.

Theorem 5.4 (Gelfand and Naimark). *If A is a commutative unital C^*-algebra, then the Gelfand transform is a $*$-isomorphism of A onto $C(M(A))$.*

Central localization

The following theorem is an easy-to-use generalization of Theorem 5.4 to non-commutative C^*-algebras. The *center* of a C^*-algebra A is the set of all elements $z \in A$ satisfying $az = za$ for all $a \in A$. Note that the center and every C^*-subalgebra of the center are commutative C^*-subalgebras of A.

Theorem 5.5 (Allan and Douglas). *Let A be a unital C^*-algebra with the unit e and let Z be a C^*-subalgebra of the center of A which contains e. For each maximal ideal $m \in M(Z)$, let J_m be the smallest closed ideal of A which contains the set m. Then an element $a \in A$ is invertible in A if and only if the elements $a + J_m \in A/J_m$ are invertible in A/J_m for all $m \in M(A)$.*

We remark that we consider $a + J_m$ as invertible in A/J_m if $J_m = A$.

The C^*-algebras A/J_m are referred to as local algebras, the spectrum of $a + J_m$ is called the local spectrum of a at m, and every element $a_m \in A$ for which

$$a_m + J_m = a + J_m$$

is said to be a local representative of a at m. Theorem 5.5 is a so-called local principle.

If A itself is commutative, we can take $Z = A$, and since then

$$A/J_m = A/m \to \mathbf{C}, \quad a + m \mapsto \varphi(a) \quad (m = \operatorname{Ker} \varphi)$$

is a $*$-isomorphism (Gelfand-Mazur theorem), Theorem 5.5 gives the same conclusion as Theorem 5.4: an element $a \in A$ is invertible if and only if $\varphi(a) \neq 0$ for all $\varphi \in M(A)$. Clearly, the larger the center of a C^*-algebra is, the finer we can localize using Theorem 5.5. In case the center is trivial, that is, equal to $\{\lambda e : \lambda \in \mathbf{C}\}$, Theorem 5.5 merely says that a is invertible if and only if a is invertible.

Lifting of ideals

The following theorem is concerned with the so-called lifting of an ideal J, that is, with conditions under which the invertibility of $a \in A$ can be deduced from the invertibility of a modulo an ideal J.

Theorem 5.6 (Roch and Silbermann). *Let A be a unital C^*-algebra and let $\{J_t\}_{t\in T}$ be a family of closed ideals of A. Denote by J the smallest closed ideal of A which contains all the ideals J_t $(t \in T)$. Suppose for each $t \in T$ we are given a unital C^*-algebra B_t and a $*$-homomorphism $\psi_t : A \to B_t$ whose restriction $\psi_t|J_t$ to J_t is injective. Then an element $a \in A$ is invertible in A if and only if $a + J$ is invertible in A/J and $\psi_t(a)$ is invertible in B_t for every $t \in T$.*

Proofs of this theorem are in [51] and [34, Theorem 5.26].

Irreducible representations

Let A be a C^*-algebra. A *representation* of A is a pair (H, π) of a Hilbert space H and a $*$-homomorphism $\pi : A \to \mathcal{B}(H)$. A closed subspace K of H is said to be invariant for a representation (H, π) if $\pi(a)K \subset K$ for all $a \in A$. A representation (H, π) is called *irreducible* if π is not the zero map and if $\{0\}$ and H are the only closed subspaces of H which are invariant for (H, π). Two irreducible representations (H_1, π_1) and (H_2, π_2) are unitarily equivalent if there exists a unitary operator $U : H_1 \to H_2$ such that

$$\pi_2(a) = U\pi_1(a)U^{-1} \text{ for all } a \in A.$$

Unitary equivalence is an equivalence relation in the set of all irreducible representations of A. The set of the equivalence classes of the irreducible representations of A with respect to unitary equivalence is referred to as the *spectrum* (or the *structure space*) of A, and we denote this set by $\operatorname{Spec} A$. In what follows we denote the equivalence class of $\operatorname{Spec} A$ which contains the representation (H, π) simply by (H, π).

Theorem 5.7. *Let A be a unital C^*-algebra and let $\{(H_t, \pi_t)\}_{t \in \operatorname{Spec} A}$ be a family of irreducible representations of A which contains one element from each equivalence class of* $\operatorname{Spec} A$*. Then an element $a \in A$ is invertible if and only if $\pi_t(a)$ is invertible in $\mathcal{B}(H_t)$ for every $t \in \operatorname{Spec} A$.*

Spectrum of ideals and quotients

Given a C^*-algebra A and a closed ideal J of A, put

$$\begin{aligned} \operatorname{Spec}_J A &= \Big\{(H, \pi) \in \operatorname{Spec} A : \ \pi(J) = \{0\}\Big\}, \\ \operatorname{Spec}^J A &= \Big\{(H, \pi) \in \operatorname{Spec} A : \ \pi(J) \neq \{0\}\Big\}. \end{aligned}$$

Obviously,

$$\operatorname{Spec} A = \operatorname{Spec}_J A \cup \operatorname{Spec}^J A, \quad \operatorname{Spec}_J A \cap \operatorname{Spec}^J A = \emptyset. \tag{17}$$

If $(H, \pi) \in \operatorname{Spec}_J A$, then the map

$$\pi/J : A/J \to \mathcal{B}(H), \ \ a + J \mapsto \pi(a)$$

is a well-defined $*$-homomorphism, and if $(H, \pi) \in \operatorname{Spec}^J A$, then the map

$$\pi|J : J \to \mathcal{B}(H), \ \ j \mapsto \pi(j)$$

is also a $*$-homomorphism.

Theorem 5.8. *Let J be a closed ideal of a C^*-algebra A. Then the maps*

$$\operatorname{Spec}_J A \to \operatorname{Spec}(A/J), \quad (H, \pi) \mapsto (H, \pi/J)$$

and

$$\operatorname{Spec}^J A \to \operatorname{Spec} J, \quad (H, \pi) \mapsto (J, \pi|J)$$

are bijections.

Here is a useful result on the spectrum of the sum of two closed ideals (recall Proposition 5.2).

Theorem 5.9. *Let A be a C^*-algebra and let J_1, J_2 be closed ideals. Then*

$$\operatorname{Spec}(J_1 + J_2) = \operatorname{Spec} J_1 \cup \operatorname{Spec} J_2.$$

Moreover, if $J_1 \cap J_2 = \{0\}$, *then*

$$\operatorname{Spec} J_1 \cap \operatorname{Spec} J_2 = \emptyset.$$

Note that in Theorem 5.9 the elements of $\operatorname{Spec} J_k$ are identified with elements of $\operatorname{Spec}^{J_k} A$ as in Theorem 5.8.

There are two simple cases in which the spectrum of a C^*-algebra can be easily described.

Theorem 5.10. (a) *If A is a commutative unital C^*-algebra, then* $\operatorname{Spec} A$ *can be identified with the set $M(A)$: every multiplicative linear functional $\varphi : A \to \mathbf{C} = \mathcal{B}(\mathbf{C})$ is an irreducible representation of A, and every irreducible representation of A is unitarily equivalent to exactly one multiplicative linear functional of A.*

(b) *If H is a Hilbert space, then the spectrum of the C^*-algebra $\mathcal{K}(H)$ is a singleton: every irreducible representation of $\mathcal{K}(H)$ is unitarily equivalent to the identical representation* $\mathrm{id} : \mathcal{K}(H) \to \mathcal{B}(H)$, $a \mapsto a$.

6. Toeplitz algebras

Given a C^*-subalgebra B of L^∞, we denote by $\mathcal{A}(B)$ the smallest C^*-subalgebra of $\mathcal{B}(l^2)$ which contains all Toeplitz operators with symbols in B. Because $\lambda T(a) = T(\lambda a)$ and $T(a) + T(b) = T(a + b)$ for $a, b \in L^\infty$ and $\lambda \in \mathbf{C}$, the C^*-algebra $\mathcal{A}(B)$ is the closure in $\mathcal{B}(l^2)$ of the set of all operators of the form

$$\sum_j \prod_k T(b_{jk}), \quad b_{jk} \in B,$$

the sum and the products finite. We are interested in the structure of the C^*-algebras $\mathcal{A}(C)$ and $\mathcal{A}(PC)$, which correspond to the continuous and piecewise continuous symbols.

Given $c \in L^\infty$, we define $\widetilde{c} \in L^\infty$ by $\widetilde{c}(t) := c(1/t)\,(t \in \mathbf{T})$. In terms of Fourier series we have

$$c(t) = \sum_{n \in \mathbf{Z}} c_n t^n, \quad \widetilde{c}(t) = \sum_{n \in \mathbf{Z}} c_{-n} t^n \quad (t \in \mathbf{T}).$$

The *Hankel operator* $H(c)$ generated by $c \in L^\infty$ is the bounded operator on l^2 given by the matrix

$$H(c) := (c_{j+k-1})_{j,k=1}^{\infty} = \begin{pmatrix} c_1 & c_2 & c_3 & \dots \\ c_2 & c_3 & \dots & \\ c_3 & \dots & & \\ \dots & & & \end{pmatrix}.$$

Obviously,

$$H(\widetilde{c}) := (c_{-j-k+1})_{j,k=1}^{\infty} = \begin{pmatrix} c_{-1} & c_{-2} & c_{-3} & \dots \\ c_{-2} & c_{-3} & \dots & \\ c_{-3} & \dots & & \\ \dots & & & \end{pmatrix}.$$

With these notations, we have the following nice formula for the product of two Toeplitz matrices:

$$T(a)T(b) = T(ab) - H(a)H(\widetilde{b}). \tag{18}$$

This formula can be easily verified by computing the corresponding entries of the matrices of each side.

Proposition 6.1. *If $c \in C$, then $H(c)$ and $H(\widetilde{c})$ are compact operators.*

Proof. This can be shown by taking into account that $\|H(c)\| \le \|c\|_\infty$ and $\|H(\widetilde{c})\| \le \|c\|_\infty$ and by approximating c by trigonometric polynomials, which induce finite-rank Hankel matrices. ■

Continuous symbols. It is easy to check that $\mathcal{A}(C)$ contains all rank-one operators and hence the set $\mathcal{K} := \mathcal{K}(l^2)$ of all compact operators (see, e.g., [18, p. 155]). From formula (18) and Proposition 6.1 we infer that if $a, b \in C$, then

$$T(a)T(b) = T(ab) + \text{ compact operator}, \tag{19}$$

and hence $\mathcal{A}(C)$ is the closure in $\mathcal{B}(l^2)$ of the set

$$\Big\{T(c) + K :\ c \in C,\ K \in \mathcal{K}\Big\}. \tag{20}$$

But the set (20) is closed. Indeed, the map

$$C \to \mathcal{B}(l^2)/\mathcal{K}, \quad c \mapsto T(c) + \mathcal{K}$$

is a $*$-homomorphism by virtue of (19), and hence its image, the set $\{T(c)+\mathcal{K} : c \in C\}$, is closed due to Proposition 5.3(b). As the map

$$\mathcal{B}(l^2) \to \mathcal{B}(l^2)/\mathcal{K}, \quad A \mapsto A + \mathcal{K}$$

is continuous, the pre-image of $\{T(c)+\mathcal{K} : c \in C\}$, that is, the set (20), must also be closed. In summary,

$$\mathcal{A}(C) = \Big\{T(c) + K : \ c \in C, \ K \in \mathcal{K}\Big\}. \tag{21}$$

Theorem 6.2. *The C^*-algebra $\mathcal{A}(C)/\mathcal{K}$ is commutative, its maximal ideal space can be identified with* $\mathbf{T}$, *and the Gelfand transform is given by*

$$\Gamma : \ \mathcal{A}(C)/\mathcal{K} \to C(\mathbf{T}), \quad \big(\Gamma\big(T(a)+\mathcal{K}\big)\big)(t) = a(t).$$

Proof. The commutativity of $\mathcal{A}(C)/\mathcal{K}$ results from (19) and (21). The map

$$C \to \mathcal{A}(C)/\mathcal{K}, \quad c \mapsto T(c) + \mathcal{K}$$

is a $*$-homomorphism due to (19), it is surjective by virtue of (21), and it is injective because of Proposition 4.2(a). Hence, this map is a $*$-isomorphism. ∎

Piecewise continuous symbols. The C^*-algebra $\mathcal{A}(PC)$ also contains $\mathcal{K}$, but as (19) is in general no longer true if a and b have jumps, we have no such simple description of $\mathcal{A}(PC)$ as in (21).

Proposition 6.3. *The C^*-algebra $\mathcal{A}(PC)/\mathcal{K}$ is commutative.*

Proof. We have to show that if $a, b \in PC$, then

$$T(a)T(b) - T(b)T(a) \in \mathcal{K}. \tag{22}$$

It suffices to prove (22) in the case where a and b each have only one jump, at $\alpha \in \mathbf{T}$ and $\beta \in \mathbf{T}$, say. Suppose first that $\alpha \neq \beta$. There are functions $\varphi, \psi \in C$ such that

$$\varphi^2 + \psi^2 = 1, \ \varphi(\alpha) = 1, \ \varphi(\beta) = 0, \ \psi(\alpha) = 0, \ \psi(\beta) = 1.$$

We have

$$\begin{aligned} &T(ab) - T(a)T(b) \\ &= T(a\varphi^2 b) + T(a\psi^2 b) - T(a)\Big(T(\varphi^2) + T(\psi^2)\Big)T(b). \end{aligned} \tag{23}$$

By (18) and Proposition 6.1,

$$\begin{aligned} &T(a\varphi^2 b) - T(a)T(\varphi^2)T(b) \\ &= T(a\varphi\varphi b) - T(a\varphi)T(\varphi b) + K \\ &= H(a\varphi)H(\widetilde{\varphi b}) + K \end{aligned} \tag{24}$$

with $K \in \mathcal{K}$. Since $\varphi b \in C$, it follows from Proposition 6.1 that (24) is compact. Analogously one can show that

$$T(a\psi^2 b) - T(a)T(\psi^2)T(b)$$

is compact. This implies that (23) is compact and therefore proves (22). If $\alpha = \beta$, there are $\lambda \in \mathbf{C}$ and $c \in C$ such that $a = \lambda b + c$. Hence, by (18),

$$\begin{aligned} &T(a)T(b) - T(b)T(a) = T(c)T(b) - T(b)T(c) \\ &= T(cb) - H(c)H(\widetilde{b}) - T(bc) + H(b)H(\widetilde{c}), \end{aligned}$$

and the compactness of $H(c)$ and $H(\widetilde{c})$ gives (21). ■

Theorem 6.4 (Gohberg and Krupnik 1969). *The C^*-algebra $\mathcal{A}(PC)/\mathcal{K}$ is commutative, its maximal ideal space can be identified with the cylinder $\mathbf{T} \times [0,1]$ (with an exotic topology), and the Gelfand transform*

$$\Gamma : \mathcal{A}(PC)/\mathcal{K} \to C\big(\mathbf{T} \times [0,1]\big)$$

acts on the generating elements $T(a) + \mathcal{K}\,(a \in PC)$ by the rule

$$\Big(\Gamma\big(T(a) + \mathcal{K}\big)\Big)(t, \mu) = a(t-0)(1-\mu) + a(t+0)\mu.$$

Proof. The commutativity of $\mathcal{A}(PC)/\mathcal{K}$ was established in Proposition 6.3.

Let φ be a multiplicative linear functional of $\mathcal{A}(PC)/\mathcal{K}$. Since the restriction of φ to $\mathcal{A}(C)/\mathcal{K}$ is a multiplicative linear functional of $\mathcal{A}(C)/\mathcal{K}$, Theorem 6.2 implies that there is a $t \in \mathbf{T}$ such that

$$\varphi\big(T(c) + \mathcal{K}\big) = c(t) \ \text{ for all } \ c \in C. \tag{25}$$

Let χ_t be the characteristic function of the arc $\{te^{i\theta} : 0 \le \theta \le \pi/2\}$. Since

$$\operatorname{sp}\big(T(\chi_t) + \mathcal{K}\big) = \operatorname{sp}_{\text{ess}} T(\chi_t) = [0,1] \tag{26}$$

by Theorem 4.5, we deduce from Theorem 5.4 that there is a $\mu \in [0,1]$ such that

$$\varphi\big(T(\chi_t) + \mathcal{K}\big) = \mu. \tag{27}$$

Every $a \in PC$ can be written in the form

$$a = a(t-0)(1-\chi_t) + a(t+0)\chi_t + cb \tag{28}$$

with $c \in C$, $c(t) = 0$, $b \in PC$. As

$$T(cb) = T(c)T(b) + H(c)H(\widetilde{b}) \in T(c)T(b) + \mathcal{K}$$

by (18) and Proposition 6.1, we obtain from (25), (27), and (28) that

$$\begin{aligned} \varphi\big(T(a)+\mathcal{K}\big) &= a(t-0)(1-\mu) + a(t+0)\mu + c(t)\varphi\big(T(b)+\mathcal{K}\big) \\ &= a(t-0)(1-\mu) + a(t+0)\mu. \end{aligned} \tag{29}$$

Thus, every multiplicative linear functional φ is of the form (29) for some (t,μ) in the cylinder $\mathbf{T} \times [0,1]$.

Conversely, suppose we are given $(t,\mu) \in \mathbf{T} \times [0,1]$. Let now χ_t be any function which is continuous on $\mathbf{T} \setminus \{t\}$ and satisfies

$$\chi_t(t-0) = 0, \quad \chi_t(t+0) = 1, \quad \chi_t(\mathbf{T} \setminus \{t\}) \cap [0,1] = \emptyset.$$

From Theorems 4.5 and 5.4 we deduce that there exists a multiplicative linear functional φ such that (27) holds. The restriction of φ to $\mathcal{A}(C)/\mathcal{K}$ must be of the form

$$\varphi\big(T(c)+\mathcal{K}\big) = c(\tau)$$

for some $\tau \in \mathbf{T}$. Suppose $\tau \neq t$. If $c \in C$ is any function such that $c(t) = 0$ and $c(\tau) \neq 0$, then $c\chi_t \in C$ and consequently,

$$c(\tau)\chi_t(\tau) = \varphi\big(T(c\chi_t)+\mathcal{K}\big) = \varphi\big(T(c)T(\chi_t)+\mathcal{K}\big) = c(\tau)\mu.$$

Since $\chi_t(\tau) \notin [0,1]$, this is impossible. Thus, $\tau = t$, which implies that (25) holds. But having (25) and (27) at our disposal, we see as in the preceding paragraph that φ satisfies (29) for all $a \in PC$. ■

Combining (17) and Theorems 5.8, 5.10, 6.2, 6.4 we arrive at the conclusion that

$$\begin{aligned} &\operatorname{Spec} \mathcal{A}(C) \cong \mathbf{T} \cup \{\mathrm{id}\}, \\ &\operatorname{Spec} \mathcal{A}(PC) \cong \big(\mathbf{T} \times [0,1]\big) \cup \{\mathrm{id}\}. \end{aligned}$$

7. Algebraization of stability

Let $\mathbf{S}$ be the set of all sequences $\{A_n\}_{n=1}^{\infty}$ of operators $A_n \in \mathcal{B}(\mathbf{C}^n)$ (equivalently, of complex $n \times n$ matrices A_n) satisfying

$$\sup_{n\geq 1} \|A_n\| < \infty.$$

In what follows we abbreviate $\{A_n\}_{n=1}^{\infty}$ to $\{A_n\}$. The set $\mathbf{S}$ is a C^*-algebra with the operations

$$\begin{aligned} \{A_n\} + \{B_n\} &:= \{A_n + B_n\}, \quad \lambda\{A_n\} := \{\lambda A_n\} \quad (\lambda \in \mathbf{C}), \\ \{A_n\}\{B_n\} &:= \{A_n B_n\}, \qquad \{A_n\}^* := \{A_n^*\} \end{aligned}$$

and the norm

$$\|\{A_n\}\| := \sup_{n\geq 1} \|A_n\|.$$

An element $\{A_n\} \in \mathbf{S}$ is invertible in $\mathbf{S}$ if and only if the matrices A_n are invertible for all $n \geq 1$ and $\sup_{n\geq 1} \|A_n^{-1}\| < \infty$. This is much more than stability.

Now denote by $\mathbf{N}$ (the $\mathbf{N}$ is for "null") the set of all sequences $\{A_n\} \in \mathbf{S}$ for which

$$\lim_{n\to\infty} \|A_n\| = 0.$$

In other terminology, $\mathbf{S}$ is the l^∞-direct sum (= direct product) and $\mathbf{N}$ is the c_0-direct sum (= direct sum) of the family $\{\mathcal{B}(\mathbf{C}^n)\}_{n=1}^{\infty}$.

Obviously, $\mathbf{N}$ is a closed ideal of $\mathbf{S}$. If $\{A_n\} \in \mathbf{S}$, then $\{A_n\} + \mathbf{N}$ is invertible in the quotient algebra $\mathbf{S}/\mathbf{N}$ if and only if there exists a $\{B_n\} \in \mathbf{S}$ such that

$$\begin{aligned} A_n B_n &= P_n + C_n', \quad \|C_n'\| \to 0, \\ B_n A_n &= P_n + C_n'', \quad \|C_n''\| \to 0, \end{aligned}$$

and this is easily seen to be equivalent to the existence of an $n_0 \geq 1$ such that A_n is invertible for all $n \geq n_0$ and

$$\sup_{n\geq n_0} \|A_n^{-1}\| < \infty.$$

Thus, $\{A_n\} + \mathbf{N}$ is invertible in $\mathbf{S}/\mathbf{N}$ if and only if $\{A_n\}$ is stable.

At this point we have rephrased the question of stability in terms of invertibility in some C^*-algebra, and we could now have recourse to the tools of Section 5. However the algebra $\mathbf{S}/\mathbf{N}$ is too large and therefore useful results (which would be results for *every* approximating sequence $\{A_n\} \in \mathbf{S}$) cannot be obtained in this way.

So let us replace $\mathbf{S}$ by a smaller sequence algebra which, however, contains all the sequences we are interested in. For a C^*-subalgebra B of L^∞ (e.g., $B = C$ or $B = PC$), let $\mathbf{S}(B)$ denote the smallest C^*-subalgebra of $\mathbf{S}$ which contains all sequences of the form $\{T_n(b)\}_{n=1}^\infty$ with $b \in \mathcal{B}$. Thus, $\mathbf{S}(B)$ is the closure in $\mathbf{S}$ of the collection of all elements

$$\sum_j \prod_k \Big\{T_n(b_{jk})\Big\}_{n=1}^\infty \text{ with } b_{jk} \in B, \tag{30}$$

the sum and the products finite. One can show that $\mathbf{N} \subset \mathbf{S}(C)$ (see [18, p. 293]) and hence $\mathbf{N} \subset \mathbf{S}(B)$ whenever $C \subset B$. By Proposition 5.1, a sequence $\{A_n\} \in \mathbf{S}(B)$ $(C \subset B)$ is stable if and only if $\{A_n\} + \mathbf{N}$ is invertible in $\mathbf{S}(B)/\mathbf{N}$.

In the following two sections we will see that the C^*-algebras $\mathbf{S}(C)/\mathbf{N}$ and $\mathbf{S}(PC)/\mathbf{N}$ can be described fairly precisely.

8. The Silbermann ideals

Suppose B is a C^*-subalgebra of L^∞ which contains C. In order to understand the C^*-algebra $\mathbf{S}(B)/\mathbf{N}$, we look for closed ideals of this algebra.

The analogue of formula (18) for finite Toeplitz matrices is due to Widom [64] and it is

$$T_n(a)T_n(b) = T_n(ab) - P_nH(a)H(\widetilde{b})P_n - W_nH(\widetilde{a})H(b)W_n. \tag{31}$$

Here we encounter the Hankel operators introduced in Section 6, and the operators W_n are given by

$$W_n : l^2 \to l^2, \ \{x_1, x_2, x_3, \dots\} \mapsto \{x_n, x_{n-1}, \dots, x_1, 0, 0, \dots\}.$$

In what follows we freely identify the image of W_n with $\mathbf{C}^n$ and W_n itself with the $n \times n$ matrix

$$\begin{pmatrix} 0 & \dots & 0 & 1 \\ 0 & \dots & 1 & 0 \\ \vdots & & \vdots & \vdots \\ 1 & \dots & 0 & 0 \end{pmatrix}.$$

Notice that $W_n^2 = P_n$, that $W_n \to 0$ weakly, and that

$$W_n T_n(a) W_n = T_n(\widetilde{a}), \tag{32}$$

where, as in Section 6, $\widetilde{a}(t) := a(1/t)\,(t \in \mathbf{T})$. We remark that all the following constructions have their roots in formula (31).

It can be shown that if K and L are compact operators, then the sequences $\{P_n K P_n\}$ and $\{W_n L W_n\}$ belong to $\mathbf{S}(C)$ (see [18, p. 293]). Thus, we can consider the two subsets

$$\mathbf{J}_0 := \Big\{\{P_n K P_n\} + \mathbf{N} :\ K \in \mathcal{K}\Big\}, \quad \mathbf{J}_1 := \Big\{\{W_n L W_n\} + \mathbf{N} :\ L \in \mathcal{K}\Big\}$$

of the C^*-algebra $\mathbf{S}(B)/\mathbf{N}$.

Theorem 8.1 (Silbermann 1981). *The sets* $\mathbf{J}_0$ *and* $\mathbf{J}_1$ *are closed ideals of* $\mathbf{S}(B)/\mathbf{N}$ *and the maps*

$$\begin{aligned} \varphi_0 :\ \mathcal{K} \to \mathbf{J}_0, &\quad K \mapsto \{P_n K P_n\} + \mathbf{N}, \\ \varphi_1 :\ \mathcal{K} \to \mathbf{J}_1, &\quad L \mapsto \{W_n L W_n\} + \mathbf{N} \end{aligned}$$

are $*$-isomorphisms.

Proof. Let $Q_n = I - P_n$. We have

$$P_n K P_n P_n L P_n = P_n K L P_n - P_n K Q_n L P_n.$$

As $Q_n \to 0$ strongly, it follows that $\|Q_n L\| \to 0$ whenever L is compact. Hence

$$\{P_n K P_n\}\{P_n L P_n\} - \{P_n K L P_n\} \in \mathbf{N}$$

for all $K, L \in \mathcal{K}$. This shows that φ_1 is a $*$-homomorphism. Consequently, by Proposition 5.3(b), $\mathbf{J}_0 = \varphi_0(\mathcal{K})$ is a C^*-subalgebra of $\mathbf{S}(B)/\mathbf{N}$. Since $\|P_n K P_n\| \to \|K\|$, the map φ_0 is injective. Thus, φ_0 is a $*$-isomorphism.

Analogously, the map φ_1 is a $*$-homomorphism, because

$$W_n K W_n W_n L W_n = W_n K P_n L W_n = W_n K L W_n - W_n K Q_n L W_n$$

and $\|Q_n L\| \to 0$ if L is compact. It follows as above that $\mathbf{J}_1 = \varphi_1(\mathcal{K})$ is a C^*-subalgebra of $\mathbf{S}(B)/\mathbf{N}$. The injectivity of φ_1 results from the observation that if L is compact and $\|W_n L W_n\| \to 0$, then

$$\|P_n L P_n\| = \|W_n W_n L W_n W_n\| \le \|W_n L W_n\| = o(1),$$

whence $\|L\| = \lim \|P_n L P_n\| = 0$. The map φ_1 is therefore also a $*$-isomorphism.

To prove that $\mathbf{J}_0$ and $\mathbf{J}_1$ are ideals, let $K \in \mathcal{K}$ and $a \in B$. We have

$$P_n K P_n T_n(a) = P_n K P_n T(a) P_n = P_n K T(a) P_n - P_n K Q_n T(a) P_n,$$

and as $KT(a) \in \mathcal{K}$ and $\|KQ_n\| \to 0$, we see that $\mathbf{J}_0$ is a left ideal of $\mathbf{S}(B)/\mathbf{N}$. Analogously one can verify that $\mathbf{J}_0$ is a right ideal of $\mathbf{S}(B)/\mathbf{N}$. That $\mathbf{J}_1$ is an ideal can be proved similarly using that

$$\begin{aligned} &W_n L W_n T_n(a) = W_n L W_n T(a) P_n = W_n L W_n T(a) W_n W_n \\ &= W_n L P_n T(\widetilde{a}) W_n = W_n L T(\widetilde{a}) W_n - W_n L Q_n T(\widetilde{a}) W_n \end{aligned}$$

(recall (32)). ■

Proposition 5.2 now implies that

$$\mathbf{J} := \mathbf{J}_0 + \mathbf{J}_1 = \Big\{ \{P_n K P_n + W_n L W_n\} + \mathbf{N} : \ K, L \in \mathcal{K} \Big\}$$

is also a closed ideal of $\mathbf{S}(B)/\mathbf{N}$. Thus, we can consider the C^*-algebra $(\mathbf{S}(B)/\mathbf{N})/\mathbf{J}$. Before proceeding its investigation, we remark that this C^*-algebra may be interpreted in a slightly different manner. We define the subset $\mathbf{I}$ of $\mathbf{S}(B)$ as

$$\Big\{ \{P_n K P_n + W_n L W_n + C_n\} : \ K \in \mathcal{K}, \ L \in \mathcal{K}, \ \{C_n\} \in \mathbf{N} \Big\}. \quad (33)$$

Proposition 8.2. *The set* $\mathbf{I}$ *is a closed ideal of* $\mathbf{S}(B)$ *and the map*

$$\mathbf{S}(B)/\mathbf{I} \to (\mathbf{S}(B)/\mathbf{N})/\mathbf{J}, \quad \{A_n\} + \mathbf{I} \mapsto (\{A_n\} + \mathbf{N}) + \mathbf{J}$$

is a well-defined $*$-isomorphism.

Proof. The canonical homomorphism

$$\mathbf{S}(B) \to (\mathbf{S}(B)/\mathbf{N})/\mathbf{J}, \quad \{A_n\} \mapsto (\{A_n\} + \mathbf{N}) + \mathbf{J}$$

is a $*$-homomorphism. Its kernel is just $\mathbf{I}$, which gives all assertions. ■

We are now in a position to reveal the structure of the elements of $\mathbf{S}(C)$.

Theorem 8.3. *The C^*-algebra $\mathbf{S}(C)$ coincides with the set of all sequences of the form*

$$\Big\{T_n(c) + P_n K P_n + W_n L W_n + C_n\Big\} \tag{34}$$

with $c \in C$, $K \in \mathcal{K}$, $L \in \mathcal{K}$, $\{C_n\} \in \mathbf{N}$.

Proof. From (31), Proposition 6.1, and Proposition 8.2 we infer that $\mathbf{S}(C)$ is the closure of the set of all elements of the form (34). We are therefore left with showing that this set is closed. The map

$$C \to \mathbf{S}(C)/\mathbf{I}, \quad c \mapsto \{T_n(c)\} + \mathbf{I}$$

is a $*$-homomorphism by virtue of (31) and Proposition 6.1. Consequently, by Proposition 5.3(b),

$$\Big\{\{T_n(c)\} + \mathbf{I} : \ c \in C\Big\} \tag{35}$$

is a C^*-subalgebra and thus a closed subset of $\mathbf{S}(C)/\mathbf{I}$. Its pre-image under the (continuous) canonical homomorphism $\mathbf{S}(C) \to \mathbf{S}(C)/\mathbf{I}$ is the set of all elements of the form (34). Hence, this set is also closed. ■

Now we are ready for the "sequence analogues" of Theorems 6.2 and 6.4.

Theorem 8.4. *The C^*-algebra $(\mathbf{S}(C)/\mathbf{N})/\mathbf{J}$ is commutative, its maximal ideal space can be identified with $\mathbf{T}$, and the Gelfand transform is given by*

$$\bigg(\Gamma\Big(\Big(\{T_n(c)\} + \mathbf{N}\Big) + \mathbf{J}\Big)\bigg)(t) = c(t) \quad (c \in C).$$

Proof. From (31) and Proposition 6.1 we infer that the algebra is commutative. The map

$$C \to \Big(\mathbf{S}(C)/\mathbf{N}\Big)/\mathbf{J}, \quad c \mapsto \Big(\{T_n(c)\} + \mathbf{N}\Big) + \mathbf{J} \tag{36}$$

is a $*$-homomorphism (again (31) and Proposition 6.1), which is surjective due to Theorem 8.3. The kernel of (36) consists of the functions $c \in C$ for which

$$T_n(c) = P_n K P_n + W_n L W_n + C_n \tag{37}$$

with certain $K, L \in \mathcal{K}$ and $\{C_n\} \in \mathbf{N}$. As $P_n K P_n \to K$ in the norm and $W_n L W_n \to 0$ strongly (note that L is compact and $W_n \to 0$ weakly), passage to the strong limit $n \to \infty$ in (37) gives $T(c) = K$, whence $c = 0$ due to Proposition 4.2(a). Thus, (36) is injective. This proves that $(\mathbf{S}(C)/\mathbf{N})/\mathbf{J}$ is $*$-isomorphic to $C = C(\mathbf{T})$. ■

Theorem 8.5. *The C^*-algebra $(\mathbf{S}(PC)/\mathbf{N})/\mathbf{J}$ is commutative, its maximal ideal space can be identified with the cylinder $\mathbf{T} \times [0,1]$ (with the same exotic topology as in Theorem* 6.4), *and the Gelfand transform acts at the generating elements by the rule*

$$\left(\Gamma\left(\{T_n(a)\} + \mathbf{N}\right) + \mathbf{J}\right)(t, \mu) = a(t-0)(1-\mu) + a(t+0)\mu.$$

Proof. Using (31) in place of (18), one can verify the commutativity of the algebra as in the proof of Proposition 6.3.

Let φ be a multiplicative linear functional. From Theorem 8.4 we infer that

$$\varphi\left(\left(\{T_n(c)\} + \mathbf{N}\right) + \mathbf{J}\right) = c(t) \text{ for all } c \in C \tag{38}$$

with some $t \in \mathbf{T}$. Let χ_t be the characteristic function in the first part of the proof of Theorem 6.4. If $\lambda \notin [0,1]$, then $\mathcal{R}(\chi_t - \lambda) = \{-\lambda, 1-\lambda\}$ is contained in an open half-plane whose boundary passes through the origin. It follows that there is a $\gamma \in \mathbf{T}$ such that $\gamma\mathcal{R}(\chi_t - \lambda)$ is a subset of the open right half-plane. We can therefore find $q \in (0,1)$ such that $q\gamma\mathcal{R}(\chi_t - \lambda)$ is contained in the disk $|z-1| < r$ for some $r < 1$. Consequently, $\|I - q\gamma T_n(\chi_t - \lambda)\| \le \|1 - q\gamma(\chi_t - \lambda)\|_\infty < 1$, which implies that $T_n(\chi_t - \lambda)$ is invertible and that $\|T_n^{-1}(\chi_t - \lambda)\| < q|\gamma|/(1-r) = q/(1-r)$. Thus, $\{T_n(\chi_t - \lambda)\}$ is stable, and we have shown that

$$\operatorname{sp}\left(\left(\{T_n(\chi_t)\} + \mathbf{N}\right) + \mathbf{J}\right) \subset \operatorname{sp}\left(\{T_n(\chi_t)\} + \mathbf{N}\right) \subset [0,1]. \tag{39}$$

From (39) and Theorem 5.4 we obtain the existence of a $\mu \in [0,1]$ such that

$$\varphi\left(\left(\{T_n(\chi_t)\} + \mathbf{N}\right) + \mathbf{J}\right) = \mu. \tag{40}$$

Using (38) and (40) we can now proceed as in the proof of Theorem 6.4 to show that

$$\varphi\Big(\Big(\{T_n(a)\}+\mathbf{N}\Big)+\mathbf{J}\Big)=a(t-0)(1-\mu)+a(t+0)\mu \tag{41}$$

for all $a\in PC$.

Conversely, let $(t,\mu)\in\mathbf{T}\times[0,1]$. From Theorem 6.4 we know that there is a multiplicative linear functional η on $\mathcal{A}(PC)/\mathcal{K}$ such that

$$\eta\Big(T(a)+\mathcal{K}\Big)=a(t-0)(1-\mu)+a(t+0)\mu$$

for all $a\in PC$. Sequences in $\mathbf{S}(PC)$ have strong limits in $\mathcal{A}(PC)$. The map

$$\xi:\mathbf{S}(PC)\to\mathcal{A}(PC),\quad \{A_n\}\mapsto \operatorname{s-lim}_{n\to\infty}A_n$$

is a $*$-homomorphism. Let $\mathbf{I}$ be the ideal (33). Because $\xi(\mathbf{I})\subset\mathcal{K}$, the quotient map

$$\xi/\mathbf{I}:\mathbf{S}(PC)/\mathbf{I}\to\mathcal{A}(PC)/\mathcal{K},\quad \{A_n\}+\mathbf{I}\mapsto\xi(\{A_n\})+\mathcal{K}$$

is a well-defined $*$-homomorphism. By Proposition 8.2, we can identify $\mathbf{S}(PC)/\mathbf{I}$ and $(\mathbf{S}(PC)/\mathbf{N})/\mathbf{J}$. It follows that $\varphi:=\eta\circ\xi/\mathbf{I}$ is a multiplicative linear functional on $(\mathbf{S}(PC)/\mathbf{N})/\mathbf{J}$ such that for all $a\in PC$,

$$\varphi\Big(\Big(\{T_n(a)\}+\mathbf{N}\Big)+\mathbf{J}\Big)=\eta\Big(T(a)+\mathcal{K}\Big)=a(t-0)(1-\mu)+a(t+0)\mu.\ \blacksquare$$

9. Toeplitz sequences algebras

Theorems 8.4 and 8.5 describe the C^*-algebras

$$\Big(\mathbf{S}(C)/\mathbf{N}\Big)/\mathbf{J}\ \text{ and }\ \Big(\mathbf{S}(PC)/\mathbf{N}\Big)/\mathbf{J}, \tag{42}$$

but in order to study the stability of sequences in $\mathbf{S}(C)$ and $\mathbf{S}(PC)$, we need invertibility criteria in the C^*-algebras

$$\mathbf{S}(C)/\mathbf{N}\ \text{ and }\ \mathbf{S}(PC)/\mathbf{N}. \tag{43}$$

Thus, we must somehow "lift" the ideal $\mathbf{J}$. This can be managed with the help of Theorem 5.6 (the lifting theorem) or, alternatively, by determining the irreducible representations of the C^*-algebras (43) (Theorems 5.7 to 5.10).

Let $B = C$ or $B = PC$. If $\{A_n\} \in \mathbf{S}(B)$, then the two strong limits

$$A := \text{s-}\lim_{n\to\infty} A_n, \quad \widetilde{A} := \text{s-}\lim_{n\to\infty} W_n A_n W_n \tag{44}$$

exist and belong to $\mathcal{A}(B)$. Indeed, for the strong limit A this was already observed in the proof of Theorem 8.5. From (32) we see that

$$\text{s-}\lim_{n\to\infty} W_n \sum_j \prod_k T_n(a_{jk}) W_n = \text{s-}\lim_{n\to\infty} \sum_j \prod_k T_n(\widetilde{a}_{jk}) = \sum_j \prod_k T(\widetilde{a}_{jk})$$

for finite sums of finite products, and an $\varepsilon/3$-argument again gives the existence of the strong limit $\widetilde{A} \in \mathcal{A}(B)$ for all $\{A_n\} \in \mathbf{S}(B)$. Throughout what follows we assume that A and $\widetilde{A}$ are given by (44).

It is readily seen that the two maps

$$\psi_0 : \mathbf{S}(B)/\mathbf{N} \to \mathcal{A}(B), \quad \{A_n\} + \mathbf{N} \mapsto A \tag{45}$$

$$\psi_1 : \mathbf{S}(B)/\mathbf{N} \to \mathcal{A}(B), \quad \{A_n\} + \mathbf{N} \mapsto \widetilde{A} \tag{46}$$

are well-defined $*$-homomorphisms.

We begin with the question of what invertibility in the C^*-algebras (42) means.

Theorem 9.1. *Let $B = C$ or $B = PC$ and suppose $\{A_n\} \in \mathbf{S}(B)$. Then the coset $(\{A_n\} + \mathbf{N}) + \mathbf{J}$ is invertible in $(\mathbf{S}(B)/\mathbf{N})/\mathbf{J}$ if and only if the operator A is a Fredholm operator.*

Proof. If $A_n = T_n(a)$ with $a \in B$, then $A = T(a)$, and we have

$$\Gamma\Big(\big(\{T_n(a)\} + \mathbf{N}\big) + \mathbf{J}\Big) = \Gamma\big(T(a) + \mathcal{K}\big)$$

by virtue of Theorems 6.2/6.4 and 8.3/8.4. Since Γ is a $*$-homomorphism, it follows that

$$\Gamma\Big(\big(\{A_n\} + \mathbf{N}\big) + \mathbf{J}\Big) = \Gamma(A + \mathcal{K})$$

for all $\{A_n\} \in \mathbf{S}(B)$. The assertion is therefore immediate from Theorem 5.4. ■

Here is the desired invertibility criterion for the C^*-algebras (43).

Theorem 9.2. *Let $B = C$ or $B = PC$ and let $\{A_n\} \in \mathbf{S}(B)$. Then $\{A_n\} + \mathbf{N}$ is invertible in $\mathbf{S}(B)/\mathbf{N}$ if and only if the two operators A and $\widetilde{A}$ are invertible.*

First proof: via the lifting theorem. We apply Theorem 5.6 to the C^*-algebra $\mathbf{S}(B)/\mathbf{N}$ and the two ideals $\mathbf{J}_0$ and $\mathbf{J}_1$. Clearly, $\mathbf{J} = \mathbf{J}_0 + \mathbf{J}_1$ is the smallest closed ideal of $\mathbf{S}(B)/\mathbf{N}$ which contains $\mathbf{J}_0$ and $\mathbf{J}_1$. The maps ψ_0 and ψ_1 defined by (45) and (46) are $*$-homomorphisms and it is clear that $\psi_0|\mathbf{J}_0$ and $\psi_1|\mathbf{J}_1$ are injective. Hence, by Theorem 5.6, $\{A_n\} + \mathbf{N}$ is invertible in $\mathbf{S}(B)/\mathbf{N}$ if and only if

$$\psi_0\Big(\{A_n\} + \mathbf{N}\Big) = A, \quad \psi_1\Big(\{A_n\} + \mathbf{N}\Big) = \widetilde{A},$$

and $(\{A_n\} + \mathbf{N}) + \mathbf{J}$ are invertible. But Theorem 9.1 implies that $(\{A_n\} + \mathbf{N}) + \mathbf{J}$ is automatically invertible if A is invertible (and thus Fredholm). ■

Second proof: via irreducible representations. We determine the spectrum of $\mathbf{S}(B)/\mathbf{N}$. From (17) we deduce that

$$\operatorname{Spec}\Big(\mathbf{S}(B)/\mathbf{N}\Big) = \operatorname{Spec}_{\mathbf{J}}\Big(\mathbf{S}(B)/\mathbf{N}\Big) \cup \operatorname{Spec}^{\mathbf{J}}\Big(\mathbf{S}(B)/\mathbf{N}\Big)$$

is a partition of $\operatorname{Spec}(\mathbf{S}(B)/\mathbf{N})$ into two disjoint subsets. From Theorem 5.8 we get the identification

$$\operatorname{Spec}_{\mathbf{J}}\Big(\mathbf{S}(B)/\mathbf{N}\Big) \cong \operatorname{Spec}\Big((\mathbf{S}(B)/\mathbf{N})/\mathbf{J}\Big),$$

and since $(\mathbf{S}(B)/\mathbf{N})/\mathbf{J}$ is a commutative C^*-algebra whose maximal ideal space is described by Theorems 8.4 and 8.5, we infer from Theorem 5.10(a) that we have the identifications

$$\operatorname{Spec}\Big((\mathbf{S}(C)/\mathbf{N})/\mathbf{J}\Big) \cong \mathbf{T},$$
$$\operatorname{Spec}\Big((\mathbf{S}(PC)/\mathbf{N})/\mathbf{J}\Big) \cong \mathbf{T} \times [0,1].$$

Hence, we are left with $\operatorname{Spec}^{\mathbf{J}}(\mathbf{S}(B)/\mathbf{N})$. Again by Theorem 5.8, there is an identification

$$\operatorname{Spec}^{\mathbf{J}}\Big(\mathbf{S}(B)/\mathbf{N}\Big) \cong \operatorname{Spec}\mathbf{J} = \operatorname{Spec}(\mathbf{J}_0 + \mathbf{J}_1),$$

and Theorem 5.9 tells us that

$$\operatorname{Spec}(\mathbf{J}_0 + \mathbf{J}_1) = \operatorname{Spec}\mathbf{J}_0 \cup \operatorname{Spec}\mathbf{J}_1,$$

the partition being disjoint because $\mathbf{J}_0 \cap \mathbf{J}_1 = \{0\}$. Theorems 8.1 and 5.10(b) imply that $\operatorname{Spec}\mathbf{J}_0$ and $\operatorname{Spec}\mathbf{J}_1$ are singletons; irreducible representations are

$$\begin{aligned}
\tau_0 &: \ \mathbf{J}_0 \to \mathcal{B}(l^2), \quad \{P_nKP_n\} + \mathbf{N} \mapsto K,\\
\tau_1 &: \ \mathbf{J}_1 \to \mathcal{B}(l^2), \quad \{W_nLW_n\} + \mathbf{N} \mapsto L.
\end{aligned}$$

Obviously, $\tau_0 = \psi_0|\mathbf{J}_0$ and $\tau_1 = \psi_1|\mathbf{J}_1$ where ψ_0 and ψ_1 are the representations (45) and (46). Putting things together, we arrive at the partitions

$$\operatorname{Spec}\Big(\mathbf{S}(C)/\mathbf{N}\Big) \cong \mathbf{T} \cup \{\psi_0\} \cup \{\psi_1\}, \tag{47}$$

$$\operatorname{Spec}\Big(\mathbf{S}(PC)/\mathbf{N}\Big) \cong \Big(\mathbf{T} \times [0,1]\Big) \cup \{\psi_0\} \cup \{\psi_1\}. \tag{48}$$

By virtue of Theorems 5.10(a) and 8.4/8.5, we can rewrite (47) and (48) in the form

$$\operatorname{Spec}\Big(\mathbf{S}(B)/\mathbf{N}\Big) \cong M\Big(\mathcal{A}(B)/\mathcal{K}\Big) \cup \{\psi_0\} \cup \{\psi_1\}.$$

Now Theorem 5.7 shows that $\{A_n\} + \mathbf{N}$ is invertible if and only if

$$\Big(\Gamma(A + \mathcal{K})\Big)(m) \neq 0 \ \text{ for all } \ m \in M\Big(\mathcal{A}(B)/\mathcal{K}\Big) \tag{49}$$

and

$$A = \psi_0\Big(\{A_n\} + \mathbf{N}\Big), \quad \widetilde{A} = \psi_1\Big(\{A_n\} + \mathbf{N}\Big)$$

are invertible. As (49) is implied by the invertibility (or even the sole Fredholmness) of A, we arrive at the assertion. ■

Although this will not be needed in the following, we remark that (47) and (48) yield the identifications

$$\operatorname{Spec}\mathbf{S}(C) \cong \mathbf{T}\cup\{\sigma_0\}\cup\{\sigma_1\}\cup\bigcup_{n=1}^{\infty}\{\pi_n\}, \tag{50}$$

$$\operatorname{Spec}\mathbf{S}(PC) \cong \big(\mathbf{T}\times[0,1]\big)\cup\{\sigma_0\}\cup\{\sigma_1\}\cup\bigcup_{n=1}^{\infty}\{\pi_n\}. \tag{51}$$

Here $t\in\mathbf{T}$ and $(t,\mu)\in\mathbf{T}\times[0,1]$ correspond to the representations

$$\begin{aligned}
&\mathbf{S}(C)\to\mathcal{B}(\mathbf{C}), \quad \{A_n\}\mapsto\Big(\Gamma\big((\{A_n\}+\mathbf{N})+\mathbf{J}\big)\Big)(t),\\
&\mathbf{S}(PC)\to\mathcal{B}(\mathbf{C}), \quad \{A_n\}\mapsto\Big(\Gamma\big((\{A_n\}+\mathbf{N})+\mathbf{J}\big)\Big)(t)
\end{aligned}$$

where Γ is as in Theorems 8.4 and 8.5, σ_0 and σ_1 are the representations

$$\begin{aligned}
\sigma_0:\ &\mathbf{S}(B)\to\mathcal{B}(l^2), \quad \{A_n\}\mapsto A,\\
\sigma_1:\ &\mathbf{S}(B)\to\mathcal{B}(l^2), \quad \{A_n\}\mapsto \widetilde{A},
\end{aligned}$$

and π_k is the representation

$$\pi_k:\ \mathbf{S}(B)\to\mathcal{B}(\mathbf{C}^k), \quad \{A_n\}\mapsto A_k.$$

The proof of (50) and (51) follows from the decomposition

$$\begin{aligned}
\operatorname{Spec}\mathbf{S}(B) &= \operatorname{Spec}_{\mathbf{N}}\mathbf{S}(B)\cup\operatorname{Spec}^{\mathbf{N}}\mathbf{S}(B)\\
&\cong \operatorname{Spec}\big(\mathbf{S}(B)/\mathbf{N}\big)\cup\operatorname{Spec}\mathbf{N};
\end{aligned}$$

the spectrum $\operatorname{Spec}(\mathbf{S}(B)/\mathbf{N})$ is given by (47) and (48), while

$$\operatorname{Spec}\mathbf{N}\cong\bigcup_{n=1}^{\infty}\{\pi_n\}.$$

10. Stability criteria

We are now ready for the harvest.

Recall that given $\{A_n\}\in\mathbf{S}(PC)$, we let A and $\widetilde{A}$ always stand for the two operators (44):

$$A=\operatorname*{s-lim}_{n\to\infty}A_n, \quad \widetilde{A}=\operatorname*{s-lim}_{n\to\infty}W_nA_nW_n.$$

Theorem 10.1. *A sequence* $\{A_n\} \in \mathbf{S}(PC)$ *is stable if and only if the two operators* A *and* $\widetilde{A}$ *are invertible.*

Proof. This is immediate from Theorem 9.2 and from what was said in Section 6. ■

For pure Toeplitz matrices, we arrive at the following.

Corollary 10.2 (Gohberg 1967). *Let* $a \in PC$. *The sequence* $\{T_n(a)\}$ *is stable if and only if* $T(a)$ *is invertible.*

Proof. If $A_n = T_n(a)$, then $A = T(a)$ and $\widetilde{A} = T(\widetilde{a})$. As $T(\widetilde{a})$ is the transpose of $T(a)$, the operator $T(\widetilde{a})$ is invertible if and only if $T(a)$ is invertible. ■

We remark that in general $\widetilde{A}$ is not the transpose of A. For example, if $A_n = T_n(a)T_n(b)$, then

$$A = T(a)T(b), \quad \widetilde{A} = T(\widetilde{a})T(\widetilde{b}),$$

and the transpose of A is $T(\widetilde{b})T(\widetilde{a})$.

In several applications one encounters perturbed Toeplitz matrices. An especially interesting case is the one in which

$$A_n = T_n(a) + P_nKP_n + W_nLW_n \tag{52}$$

with $a \in PC$, $K \in \mathcal{K}$, $L \in \mathcal{K}$. If the matrices of K and L have only finitely many nonzero entries, then A_n results from $T_n(a)$ by constant perturbations in the upper left and lower right corners. For instance, if

$$A_n = \left(\begin{array}{cc|cccccc|cc} 2 & 9 & 2 & & & & & & & \\ 1 & 2 & 5 & 2 & & & & & & \\ \hline & 3 & 7 & 5 & 2 & & & & & \\ & & 3 & 7 & 5 & 2 & & & & \\ & & & 3 & 7 & 5 & 2 & & & \\ & & & & \dots & \dots & \dots & \dots & & \\ & & & & & & 3 & 7 & 5 & 2 \\ \hline & & & & & & & 3 & 1 & 9 \\ & & & & & & & & 5 & 4 \end{array}\right),$$

we can write A_n in the form (52) with

$$a(t) = 3t + 7 + 5t^{-1} + 2t^{-2} \quad (t \in \mathbf{T}),$$

$$K = \begin{pmatrix} -5 & 4 & 0 & \dots \\ -2 & -5 & 0 & \dots \\ 0 & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}, \quad L = \begin{pmatrix} -3 & 2 & 0 & \dots \\ 4 & -6 & 0 & \dots \\ 0 & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}.$$

Corollary 10.3 (Silbermann 1981). *Let $a \in PC$ and $K, L \in \mathcal{K}$. Then the sequence $\{T_n(a) + P_n K P_n + W_n L W_n\}$ is stable if and only if the operators $T(a) + K$ and $T(\widetilde{a}) + L$ are invertible.*

Proof. This is a straightforward consequence of Theorem 10.1. ■

In particular, the sequence $\{I + P_n K P_n + W_n L W_n\}$ (with $K, L \in \mathcal{K}$) is stable if and only if $I + K$ and $I + L$ are invertible. Clearly, given any two finite subsets M and N of $\mathbf{C}$, we can find operators K and L whose matrices have only finitely many nonzero entries and for which $\operatorname{sp}(I + K) = M \cup \{1\}$ and $\operatorname{sp}(I + L) = N \cup \{1\}$.

11. Condition numbers

Theorem 9.2 in conjunction with a simple C^*-algebra argument yields more than—the already very interesting—results of Section 10: it gives the limit of the norms $\|A_n\|$ and $\|A_n^{-1}\|$ for sequences $\{A_n\} \in \mathbf{S}(PC)$.

Theorem 11.1. *If $\{A_n\} \in \mathbf{S}(PC)$, then*

$$\lim_{n\to\infty} \|A_n\| = \max(\|A\|, \|\widetilde{A}\|).$$

Proof. The direct sum $\mathcal{B}(l^2) \oplus \mathcal{B}(l^2)$ of all ordered pairs (B, C) with $B, C \in \mathcal{B}(l^2)$ is a C^*-algebra under the norm

$$\|(B, C)\| := \max(\|B\|, \|C\|).$$

Let ψ_0 and ψ_1 be the $*$-homomorphisms (45) and (46). The map

$$\psi = \psi_0 \oplus \psi_1 : \ \mathbf{S}(PC)/\mathbf{N} \to \mathcal{B}(l^2) \oplus \mathcal{B}(l^2), \ \{A_n\} + \mathbf{N} \mapsto (A, \widetilde{A})$$

is also a $*$-homomorphism, and Theorem 9.2 tells us that ψ preserves spectra. Hence, by Proposition 5.3(e), ψ also preserves norms, which means that

$$\limsup_{n\to\infty} \|A_n\| = \max\Big(\|A\|, \|\widetilde{A}\|\Big). \tag{53}$$

As $A_n \to A$ strongly, we have

$$\|A\| \le \liminf_{n\to\infty} \|A_n\|, \tag{54}$$

and because $W_n A_n W_n \to \widetilde{A}$ strongly and $\|W_n\| = 1$, we obtain that

$$\|\widetilde{A}\| \le \liminf_{n\to\infty} \|W_n A_n W_n\| \le \liminf_{n\to\infty} \|A_n\|. \tag{55}$$

Combining (53), (54), (55), we arrive at the assertion. ■

Recall that we put $\|B^{-1}\| = \infty$ in case B is not invertible.

Theorem 11.2. *If* $\{A_n\} \in \mathbf{S}(PC)$, *then*

$$\lim_{n\to\infty} \|A_n^{-1}\| = \max\Big(\|A^{-1}\|, \|\widetilde{A}^{-1}\|\Big).$$

Proof. Suppose first that

$$\max\left(\|A^{-1}\|, \|\widetilde{A}^{-1}\|\right) = \infty.$$

Then $\{A_n\}$ is not stable by virtue of Theorem 10.1, and hence

$$\limsup_{n\to\infty} \|A_n^{-1}\| = \infty. \tag{56}$$

If there would exist a subsequence $\{n_k\}_{k=1}^{\infty}$ such that

$$\limsup_{k\to\infty} \|A_{n_k}^{-1}\| < \infty, \tag{57}$$

then also

$$\begin{aligned} \limsup_{k\to\infty} \|(W_{n_k} A_{n_k} W_{n_k})^{-1}\| &= \limsup\nolimits_{k\to\infty} \|W_{n_k} A_{n_k}^{-1} W_{n_k}\| \\ &\le \limsup\nolimits_{k\to\infty} \|A_{n_k}^{-1}\| < \infty, \end{aligned}$$

and proceeding as in the proof of Proposition 1.2, only with the natural numbers replaced by $\{n_1, n_2, n_3, \ldots\}$, it would follow that A and $\widetilde{A}$ are invertible. Thus, (57) cannot hold, and (56) therefore shows that actually $\lim \|A_n\| = \infty$.

Now suppose

$$\max\left(\|A^{-1}\|, \|\widetilde{A}^{-1}\|\right) < \infty.$$

Then $\{A_n\} + \mathbf{N}$ is invertible in the C^*-algebra $\mathbf{S}(PC)/\mathbf{N}$ due to Theorem 9.2. Let $\{B_n\} + \mathbf{N}$ be the inverse. Then $A_n^{-1} = B_n + C_n$ with $\|C_n\| \to 0$, and Theorem 11.1 gives

$$\lim_{n\to\infty} \|A_n^{-1}\| = \lim_{n\to\infty} \|B_n + C_n\| = \lim_{n\to\infty} \|B_n\| = \max(\|B\|, \|\widetilde{B}\|).$$

As, obviously, $B = A^{-1}$ and $\widetilde{B} = \widetilde{A}^{-1}$, we get the assertion. ■

Corollary 11.3. *If $a \in PC$, then*

$$\lim_{n\to\infty} \|T_n^{-1}(a)\| = \|T^{-1}(a)\|.$$

Proof. In this case $\widetilde{A} = T(\widetilde{a})$ is the transpose of $A = T(a)$, whence $\|\widetilde{A}^{-1}\| = \|A^{-1}\|$. ■

The equality $\lim \|T_n(a)\| = \|T(a)\|$ is trivial. We therefore see that if $a \in PC$, then

$$\lim_{n\to\infty} \kappa(T_n(a)) = \kappa(T(a)). \tag{58}$$

Corollary 11.4. *Let $a \in PC, K \in \mathcal{K}, L \in \mathcal{K}$, and put*

$$A_n = T_n(a) + P_nKP_n + W_nLW_n.$$

Then

$$\begin{aligned} \lim_{n\to\infty} \|A_n\| &= \max\Big(\|T(a)+K\|, \|T(\widetilde{a})+L\|\Big), \\ \lim_{n\to\infty} \|A_n^{-1}\| &= \max\Big(\|(T(a)+K)^{-1}\|, \|(T(\widetilde{a})+L)^{-1}\|\Big). \end{aligned}$$

Proof. Because $A = T(a) + K$ and $\widetilde{A} = T(\widetilde{a}) + L$, this is immediate from Theorems 11.1 and 11.2. ■

In the case where $a = 1$ identically, we encounter the operators

$$I + K, \quad I + L, \quad (I+K)^{-1}, \quad (I+L)^{-1}.$$

Let

$$\begin{aligned} K &= \operatorname{diag}\Big(0, -\frac{3}{4}, 0, 0, \ldots\Big), \\ L &= \operatorname{diag}\Big(2, -\frac{1}{2}, 0, 0, \ldots\Big). \end{aligned}$$

Then for $n \geq 4$,

$$A_n := I + P_n K P_n + W_n L W_n = \operatorname{diag}\left(1, \frac{1}{4}, \underbrace{1, \dots, 1}_{n-4}, \frac{1}{2}, 3\right).$$

We have

$$\begin{aligned}
&\|A_n\| = 3, \quad \|A_n^{-1}\| = 4, \\
&\|A\| = \|I + K\| = 1, \quad \|A^{-1}\| = \|(I + K)^{-1}\| = 4, \\
&\|\widetilde{A}\| = \|I + L\| = 3, \quad \|\widetilde{A}^{-1}\| = \|(I + L)^{-1}\| = 2,
\end{aligned}$$

and hence

$$\lim_{n\to\infty} \kappa(A_n) = 12, \quad \max\left(\kappa(A), \kappa(\widetilde{A})\right) = 6, \quad \kappa(A) = 4.$$

This shows that, in contrast to (58), for general $\{A_n\} \in \mathbf{S}(PC)$ the limit of $\kappa(A_n)$ may be larger than $\max(\kappa(A), \kappa(\widetilde{A}))$ and hence larger than $\kappa(A)$.

Moral. The reasoning of this section demonstrates the power of the C^*-algebra approach fairly convincingly. The problem of computing the limit of $\|T_n^{-1}(a)\|$ for an individual Toeplitz operator $T(a)$ is, of course, much more difficult than the question about the stability of the individual sequence $\{T_n(a)\}$. We replaced this more difficult problem for an individual operator by a simpler problem, the stability, for all sequences from a C^*-algebra, namely for the sequences of $\mathbf{S}(PC)$. Having solved the simpler problem for the C^*-algebra (Theorem 9.2), it was an easy C^*-algebra argument (Proposition 5.3(e)) which led to the solution of the more difficult problem for the given operator $T(a)$. Into the bargain we even got the solution of the more difficult problem for all sequences in $\mathbf{S}(PC)$ (Theorem 11.2). The following strategy seems to be not very promising at first glance, but it is a standard approach in operator theory, and as the present section shows, it may also be successful in numerical analysis.

$$\boxed{\text{difficult problem for an individual operator}}$$

$$\downarrow$$

$$\boxed{\text{simpler problem for a } C^*\text{-algebra}}$$

$$\downarrow$$

$$\boxed{\text{result for the } C^*\text{-algebra}}$$

$$\downarrow$$

$$\boxed{\text{desired result for the individual operator}}$$

12. Eigenvalues of Hermitian matrices

We now apply Theorem 10.1 to the problem of studying the asymptotic eigenvalue distribution of sequences of matrices. Note that, in contrast to Section 3, we now let A_n stand for an arbitrary sequence in $\mathbf{S}(PC)$ and not necessarily for sequences of the form $\{P_n A P_n\}$. We begin with a simple result for general sequences in $\mathbf{S}(PC)$.

Proposition 12.1. *If* $\{A_n\} \in \mathbf{S}(PC)$, *then*

$$\liminf_{n\to\infty} \operatorname{sp}(A_n) \subset \limsup_{n\to\infty} \operatorname{sp}(A_n) \subset \operatorname{sp} A \cup \operatorname{sp} \widetilde{A}.$$

Proof. Let $\lambda \notin \operatorname{sp} A \cup \operatorname{sp} \widetilde{A}$. Then $A - \lambda I$ and $(A-\lambda I)^{\sim} = \widetilde{A} - \lambda I$ are invertible, and hence Theorem 10.1 implies the existence of a natural number n_0 and a positive number M such that

$$\|(A_n - \lambda I)^{-1}\| < M \quad \text{for all} \quad n \ge n_0.$$

It follows that the spectral radius of $(A_n - \lambda I)^{-1}$ is less than M, which gives that

$$U_{1/M}(0) \cap \operatorname{sp}(A_n - \lambda I) = \emptyset \quad \text{for all} \quad n \ge n_0;$$

here $U_\delta(\mu) := \{\zeta \in \mathbf{C} : |\zeta - \mu| < \delta\}$. Consequently,

$$U_{1/M}(\lambda) \cap \operatorname{sp} A_n = \emptyset \quad \text{for all} \quad n \ge n_0,$$

and this shows that $\lambda \notin \limsup A_n$. ■

In the case Hermitian matrices, all inclusions of Proposition 12.1 become equalities.

Theorem 12.2. *If $\{A_n\} \in \mathbf{S}(PC)$ is a sequence of Hermitian matrices, $A_n = A_n^*$ for all n, then*

$$\liminf_{n\to\infty} \operatorname{sp} A_n = \limsup_{n\to\infty} \operatorname{sp} A_n = \operatorname{sp} A \cup \operatorname{sp} \widetilde{A}.$$

Proof. The operators A and $\widetilde{A}$ are necessarily selfadjoint, and all occurring spectra are subsets of the real line. By virtue of Proposition 12.1, it remains to show that if λ is a real number and $\lambda \notin \liminf \operatorname{sp} A_n$, then $\lambda \notin \operatorname{sp} A \cup \operatorname{sp} \widetilde{A}$. But if λ is real and not in $\liminf \operatorname{sp} A_n$, then there is a $\delta > 0$ such that

$$U_\delta(\lambda) \cap \operatorname{sp} A_{n_k} = \emptyset \text{ for infinitely many } n_k,$$

whence

$$U_\delta(0) \cap \operatorname{sp} (A_{n_k} - \lambda I) = \emptyset \text{ for infinitely many } n_k.$$

As $A_{n_k} - \lambda I$ is Hermitian, the spectral radius and the norm of $(A_{n_k} - \lambda I)^{-1}$ coincide, which gives

$$\|(A_{n_k} - \lambda I)^{-1}\| < \frac{1}{\delta} \text{ for infinitely many } n_k.$$

Thus $\{A_{n_k} - \lambda I)\}_{k=1}^\infty$ is a stable sequence. The argument of the proof of Proposition 1.2, only with the natural numbers replaced by $\{n_k\}_{k=1}^\infty$, shows that $A - \lambda I$ is invertible. As $\{W_{n_k}(A_{n_k} - \lambda I)W_{n_k}\}_{k=1}^\infty$ is stable together with $\{A_{n_k} - \lambda I\}_{k=1}^\infty$, it follows analogously that $\widetilde{A} - \lambda I$ is invertible. ■

Corollary 12.3. *If $a \in PC$ is real-valued, then*

$$\liminf_{n\to\infty} \operatorname{sp} T_n(a) = \limsup_{n\to\infty} \operatorname{sp} T_n(a) = \operatorname{sp} T(a) = [\operatorname{ess\,inf} a, \operatorname{ess\,sup} a].$$

Proof. Theorem 12.2 tells us that the limiting sets coincide with $\operatorname{sp} T(a) \cup \operatorname{sp} T(\widetilde{a})$. Because $T(\widetilde{a})$ is the transpose of $T(a)$, we have $\operatorname{sp} T(a) = \operatorname{sp} T(\widetilde{a})$. Finally, Theorem 4.5 implies that $\operatorname{sp} T(a) = [\operatorname{ess\,inf} a, \operatorname{ess\,sup} a]$. ■

Corollary 12.4. *Let $a \in PC$ be real-valued and let K and L be compact and selfadjoint operators. Put*

$$A_n = T_n(a) + P_n K P_n + W_n L W_n.$$

Then

$$\liminf_{n\to\infty} \operatorname{sp} A_n = \limsup_{n\to\infty} \operatorname{sp} A_n = \operatorname{sp}(T(a)+K) \cup \operatorname{sp}(T(\widetilde{a})+L).$$

Proof. Immediate from Theorem 12.2.. ■

Notice that, by virtue of Theorem 4.3 (or, again, in view of Theorem 4.5), $\operatorname{sp} T(a)$ is a subset of both $\operatorname{sp}(T(a)+K)$ and $\operatorname{sp}(T(\widetilde{a})+L)$.

Once more Example 3.5. Let χ be the characteristic function of the upper half-circle. The spectrum of the operator of multiplication by χ on $L^2(\mathbf{T})$ is $\mathcal{R}(\chi) = \{0,1\}$. An orthonormal basis in $L^2(\mathbf{T})$ is constituted by the function $\{e_k\}_{k=-\infty}^{\infty}$ where $e_k(t) = t^n/\sqrt{2\pi}\,(t \in \mathbf{T})$. The matrix representation of $M(\chi)$ with respect to the basis

$$\{e_0, e_1, e_{-1}, e_2, e_{-2}, e_3, e_{-3}, \ldots\}$$

is just the matrix A we encountered in Example 3.5. Thus, $\operatorname{sp} A = \{0,1\}$. On the other hand, there are permutation matrices Π_n such that $\Pi_n A_n \Pi_n = T_n(\chi)$, whence

$$\operatorname{sp} A_n = T_n(\chi).$$

From Corollary 12.3 we therefore see that the uniform and partial limiting sets of $\operatorname{sp} A_n$ are both equal to $[0,1]$. ■

13. Singular values

The *singular values* of an operator or a matrix A are the points in the spectrum of the selfadjoint and positively semi-definite operator/matrix $(A^*A)^{1/2}$. We denote the set of the singular values of A by $\Sigma(A)$.

Clearly, $(T_n^*(a)T_n(a))^{1/2}$ is in general not a Toeplitz matrix. But the sequence $\{(T_n^*(a)T_n(a))^{1/2}\}$ certainly belongs to the C^*-algebra $\mathbf{S}(PC)$, and as this sequence consists of Hermitian matrices, we can have recourse to Theorem 12.2.

Theorem 13.1. *If* $\{A_n\} \in \mathbf{S}(PC)$, *then*

$$\liminf_{n\to\infty} \Sigma(A_n) = \limsup_{n\to\infty} \Sigma(A_n) = \Sigma(A) \cup \Sigma(\widetilde{A}).$$

Proof. This follows from applying Theorem 12.2 to $\{A_n^* A_n\} \in \mathbf{S}(PC)$, which gives

$$\liminf_{n\to\infty} \mathrm{sp}\,(A_n^* A_n) = \limsup_{n\to\infty} \mathrm{sp}\,(A_n^* A_n) = \mathrm{sp}\,(A^* A) \cup \mathrm{sp}\,(\widetilde{A}^* \widetilde{A}),$$

together with the observation $\mathrm{sp}\,(B^* B) = \{\sigma^2 : \sigma \in \Sigma(B)\}$. ■

Corollary 13.2. *If* $a \in PC$, *then*

$$\liminf_{n\to\infty} \Sigma(T_n(a)) = \limsup_{n\to\infty} \Sigma(T_n(a)) = \Sigma(T(a)) \cup \Sigma(T(\widetilde{a})). \blacksquare$$

We remark that the identity

$$\Sigma(T(a)) \cup \{0\} = \Sigma(T(\widetilde{a})) \cup \{0\}$$

always holds, but in general $\Sigma(T(a))$ and $\Sigma(T(\widetilde{a}))$ may be different sets. Indeed, letting $a(t) = t$ $(t \in \mathbf{T})$, we have

$$T(a) = \begin{pmatrix} 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix},$$

$$T(\widetilde{a}) = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 1 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix},$$

so $T^*(a)T(a)$ and $T^*(\widetilde{a})T(\widetilde{a})$ are equal to

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 1 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 0 & 1 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix},$$

respectively, whence $\Sigma(T(a)) = \{1\}$ and $\Sigma(T(\widetilde{a})) = \{0, 1\}$.

14. Eigenvalues of non-Hermitian matrices

This section does not deal with the application of C^*-algebras to numerical analysis, its purpose is rather to motivate the next section, where we will again reap the harvest of the work done by C^*-algebras in Section 11.

As a matter of fact, only little is known about the limiting sets of $\operatorname{sp} T_n(a)$ for general (complex-valued) symbols $a \in PC$.

It turns out that in the case of rational symbols a the spectra $\operatorname{sp} T_n(a)$ need not at all mimic $\operatorname{sp} T(a)$. The simplest example is

$$T(a) = \begin{pmatrix} 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix},$$

where $\operatorname{sp} T(a)$ is the closed unit disk (which follows from Theorem 4.4), whereas $\operatorname{sp} T_n(a) = \{0\}$ for all n.

For the sake of simplicity, let us consider Toeplitz band matrices, that is, let us assume that a is a trigonometric polynomial:

$$a(t) = \sum_{k=-p}^{q} a_k t^k \ \ (t \in \mathbf{T}), \quad p \geq 1, \quad q \geq 1, \quad a_{-p} a_q \neq 0.$$

Given $r \in (0, \infty)$, define the trigonometric polynomial a_r by

$$a_r(t) = \sum_{k=-p}^{q} a_k r^k t^k \ (t \in \mathbf{T})$$

(note that if a would have infinitely many nonzero Fourier coefficients, then such a definition of a_r runs into problems with the convergence of the Fourier series). Since

$$T_n(a_r) = \operatorname{diag}(r, r^2, \dots, r^n)\, T_n(a)\, \operatorname{diag}(r^{-1}, r^{-2}, \dots, r^{-n}),$$

we have $\operatorname{sp} T_n(a_r) = \operatorname{sp} T_n(a)$. Thus, if there were any reason for $T_n(a)$ to mimic $\operatorname{sp} T(a)$, this reason would also force $\operatorname{sp} T_n(a)$ to mimic $\operatorname{sp} T(a_r)$. As the spectrum of $T(a)$ is determined by $a(\mathbf{T})$ and the spectrum of $T(a_r)$ is dictated by $a(r\mathbf{T})$, it is clear that in general both spectra are distinct. The equality $\operatorname{sp} T_n(a_r) = \operatorname{sp} T_n(a)$ and Proposition 12.1 give

$$\liminf_{n\to\infty} \operatorname{sp} T_n(a) \subset \limsup_{n\to\infty} \operatorname{sp} T_n(a) \subset \bigcap_{r\in(0,\infty)} \operatorname{sp} T(a_r).$$

Actually we have the following.

Theorem 14.1 (Schmidt and Spitzer 1960). *If a is a trigonometric polynomial, then*

$$\liminf_{n\to\infty} \operatorname{sp} T_n(a) = \limsup_{n\to\infty} \operatorname{sp} T_n(a) = \bigcap_{r\in(0,\infty)} \operatorname{sp} T(a_r).$$

One can show that $\bigcap_{r\in(0,\infty)} \operatorname{sp} T(a_r)$ is always connected and that it is the union of at most finitely many analytic arcs.

A result similar to Theorem 14.1 was proved in 1975 by K.M. Day for rational symbols. For non-rational symbols, rigorous results were established by Widom [66], Basor and Morrison [5], [6], Tyrtyshnikov and Zamarashkin [61], and Tilli [59], to cite only a few sample papers (also see [19]). These results are about clusters and Szegö type formulas for the eigenvalues of $T_n(a)$ for large n.

We confine ourselves to citing a pretty nice recent observation by Tilli [59]. For $a \in L^\infty$, let $\mathcal{R}(a)$ denote the essential range of a. The complement $\mathbf{C}\setminus\mathcal{R}(a)$ is the union of connected components, exactly one which, say Ω_0, is unbounded, while the other components, say $\Omega_1, \Omega_2, \ldots$, are bounded. The extended essential range $\mathcal{ER}(a)$ is defined as

$$\mathcal{ER}(a) = \mathcal{R}(a) \cup \Omega_1 \cup \Omega_2 \cup \ldots .$$

If $\mathbf{C}\setminus\mathcal{R}(a)$ has no bounded components, we put $\mathcal{ER}(a) = \mathcal{R}(a)$.

Theorem 14.2 (Tilli 1999). *For every $a \in L^\infty$, the set $\mathcal{ER}(a)$ is a cluster for the eigenvalues of $T_n(a)$, that is, if U is any open set which contains $\mathcal{ER}(a)$ and if $\gamma(U,n)$ stands for the number of eigenvalues of $T_n(a)$ in U (multiplicities taken into account), then*

$$\lim_{n\to\infty} \frac{\gamma(U,n)}{n} = 1.$$

In other words, only $o(n)$ of the n eigenvalues of $T_n(a)$ lie outside U as $n \to \infty$. If, for example, $a \in PC$ and $\mathcal{R}(a)$ is a finite set, Theorem 14.2 implies that $\liminf \operatorname{sp} T_n(a)$ contains a point of $\mathcal{R}(a)$; Tilli showed that even all of $\mathcal{R}(a)$ is contained in $\liminf \operatorname{sp} T_n(a)$.

15. Pseudospectra

Theorem 14.1 tells us that the spectra $\operatorname{sp} T_n(a)$ need not mimic $\operatorname{sp} T(a)$ as n goes to infinity. In contrast to this, pseudospectra behave as nicely as we could ever expect.

For $\varepsilon > 0$, the ε-pseudospectrum $\operatorname{sp}_\varepsilon B$ of a bounded Hilbert space operator B is defined as the set

$$\operatorname{sp}_\varepsilon B := \left\{\lambda \in \mathbf{C} : \ \|(B - \lambda I)^{-1}\| \geq \frac{1}{\varepsilon}\right\};$$

here, as usual, $\|(B - \lambda I)^{-1}\| = \infty$ if $B - \lambda I$ is not invertible. Thus $\operatorname{sp} B \subset \operatorname{sp}_\varepsilon B$ for every $\varepsilon > 0$.

In the same way the question "Is B invertible ?" is in numerical analysis better replaced by the question "What is $\|B^{-1}\|$?" (or, still better, by the question "What is $\kappa(B)$?"), the pseudospectra of matrices and operators are, in a sense, of even greater import than their usual spectra.

The following theorem provides an alternative description of pseudospectra.

Theorem 15.1. *Let A be a unital C^*-algebra and $b \in A$. Then for every $\varepsilon > 0$,*

$$\left\{\lambda \in \mathbf{C} : \ \|(b - \lambda e)^{-1}\| \geq \frac{1}{\varepsilon}\right\} = \bigcup_{\|c\| \leq \varepsilon} \operatorname{sp}(b + c),$$

the union over all $c \in A$ of norm at most ε.

Taking $A = \mathcal{B}(H)$, the C^*-algebra of all bounded linear operators on a Hilbert space H, we get

$$\operatorname{sp}_\varepsilon B = \bigcup_{\|C\| \leq \varepsilon} \operatorname{sp}(B + C). \tag{59}$$

Equality (59) is one of the reasons that the plots we see on the computer's screen are sometimes closer to pseudospectra than to the usual spectra. On the other hand, equality (59) can often be used to get an idea of the ε-pseudospectrum of a matrix B: simply perturb B randomly by (say 50) matrices C satisfying $\|C\| \leq \varepsilon$ and look at the superposition of the plots of the spectra (= eigenvalues) of $B + C$.

Clearly, to understand the behavior of $\mathrm{sp}_\varepsilon\, T_n(a)$ for large n, we need precise information about the norms $\|T_n^{-1}(a - \lambda)\|$ as $n \to \infty$. But just such kind of information is given by Theorem 11.2. Here is the result of this section.

Theorem 15.2. *If* $\{A_n\} \in \mathbf{S}(PC)$ *and* $\varepsilon > 0$, *then*

$$\liminf_{n\to\infty} \mathrm{sp}_\varepsilon\,(A_n) = \limsup_{n\to\infty} \mathrm{sp}_\varepsilon\,(A_n) = \mathrm{sp}_\varepsilon\, A \cup \mathrm{sp}_\varepsilon\, \widetilde{A}.$$

Proof. We first show that $\mathrm{sp}_\varepsilon\, A \subset \liminf \mathrm{sp}_\varepsilon\, A_n$. If $\lambda \in \mathrm{sp}\, A$, then $\|(A_n - \lambda I)^{-1}\| \to \infty$ by virtue of Theorem 11.2, which implies that λ belongs to $\liminf \mathrm{sp}_\varepsilon\, A_n$. So suppose $\lambda \in \mathrm{sp}_\varepsilon\, A \setminus \mathrm{sp}\, A$. Then $\|(A - \lambda I)^{-1}\| \geq 1/\varepsilon$. Let $U \subset \mathbf{C}$ be any open neighborhood of λ. As the norm of the resolvent of a Hilbert space operator cannot be locally constant (see, e.g., [19, Theorem 3.14]), there is a point $\mu \in U$ such that $\|(A - \mu I)^{-1}\| > 1/\varepsilon$. Hence, we can find a k_0 such that

$$\|(A - \mu I)^{-1}\| \geq \frac{1}{\varepsilon - 1/k} \quad \text{for all} \quad k \geq k_0.$$

As U was arbitrary, it follows that there exists a sequence $\lambda_1, \lambda_2, \dots$ such that $\lambda_k \in \mathrm{sp}_{\varepsilon - 1/k}\, A$ and $\lambda_k \to \lambda$. By Theorem 11.2,

$$\lim_{n\to\infty} \|(A_n - \lambda_k I)^{-1}\| \geq \frac{1}{\varepsilon - 1/k}.$$

Consequently, $\|(A_n - \lambda_k I)^{-1}\| \geq 1/\varepsilon$ and thus $\lambda_k \in \mathrm{sp}_\varepsilon\, A_n$ for all $n \geq n(k)$. This shows that $\lambda = \lim \lambda_k$ belongs to $\liminf \mathrm{sp}_\varepsilon\, A_n$.

Repeating the above reasoning with $W_n A_n W_n$ and $\widetilde{A}$ in place of A_n and A, respectively, we obtain

$$\mathrm{sp}_\varepsilon\, \widetilde{A} \subset \liminf_{n\to\infty} \mathrm{sp}_\varepsilon\, W_n A_n W_n.$$

Obviously, $\operatorname{sp}_\varepsilon W_n A_n W_n = \operatorname{sp}_\varepsilon A_n$. In summary, we have proved that

$$\operatorname{sp}_\varepsilon A \cup \operatorname{sp}_\varepsilon \widetilde{A} \subset \liminf_{n\to\infty} \operatorname{sp}_\varepsilon A_n.$$

In order to prove the inclusion $\limsup \operatorname{sp}_\varepsilon A_n \subset \operatorname{sp}_\varepsilon A \cup \operatorname{sp}_\varepsilon \widetilde{A}$, suppose λ is not in$\operatorname{sp}_\varepsilon A \cup \operatorname{sp}_\varepsilon \widetilde{A}$. Then $\|(A-\lambda I)^{-1}\| < 1/\varepsilon$ and $\|(\widetilde{A}-\lambda I)^{-1}\| < 1/\varepsilon$, whence

$$\|(A_n - \lambda I)^{-1}\| < \frac{1}{\varepsilon} - \delta < \frac{1}{\varepsilon} \ \text{ for all } \ n \geq n_0$$

with some $\delta > 0$ due to Theorem 11.2. If $n \geq n_0$ and $|\mu - \lambda| < \varepsilon\delta(1/\varepsilon - \delta)^{-1}$, then

$$\begin{aligned}\|(A_n - \mu I)^{-1}\| &\leq \frac{\|(A_n - \lambda I)^{-1}\|}{1 - |\mu - \lambda|\,\|(A_n - \lambda I)^{-1}\|}\\ &< \frac{1/\varepsilon - \delta}{1 - \varepsilon\delta(1/\varepsilon - \delta)^{-1}(1/\varepsilon - \delta)} = \frac{1}{\varepsilon},\end{aligned}$$

thus $\mu \notin \operatorname{sp}_\varepsilon A_n$. This shows that λ cannot belong to $\limsup \operatorname{sp}_\varepsilon A_n$. ■

Corollary 15.3. *If $a \in PC$, then for each $\varepsilon > 0$,*

$$\liminf_{n\to\infty} \operatorname{sp}_\varepsilon T_n(a) = \limsup_{n\to\infty} \operatorname{sp}_\varepsilon T_n(a) = \operatorname{sp}_\varepsilon T(a).$$

Proof. It is clear that $\operatorname{sp}_\varepsilon T(a) = \operatorname{sp}_\varepsilon T(\widetilde{a})$. ■

16. Asymptotic Moore-Penrose inversion

The purpose of this section is to show how a look at the C^*-algebras behind the scenes may help us to distinguish between good and no so good questions of numerical analysis.

Let B be a C^*-algebra. An element $a \in B$ is said to be *Moore-Penrose invertible* if there exists an element $b \in B$ such that

$$aba = a, \quad bab = b, \quad (ab)^* = ab, \quad (ba)^* = ba. \tag{60}$$

If (60) is satisfied for some $b \in B$, then this element b is uniquely determined. It is called the *Moore-Penrose inverse* of a and is denoted by a^+.

Theorem 16.1. *An element a of a unital C^*-algebra is Moore-Penrose invertible if and only if there exists a number $d > 0$ such that*

$$\mathrm{sp}\,(a^*a) \subset \{0\} \cup [d^2, \infty).$$

Such a result is implicit in Harte and Mbekhta's paper [35]. As far as I know, it was Roch and Silbermann [52], [53] who were the first to state the result explicitly. A full proof is also in [19, Theorem 4.21].

Corollary 16.2. *Let B be a unital C^*-algebra with the unit e and let A be a C^*-subalgebra of B which contains e. If $a \in A$ is Moore-Penrose invertible in B, then $a^+ \in A$.*

Proof. Combine Proposition 5.1 and Theorem 16.1. ∎

Now let H be a Hilbert space and consider the C^*-algebra $\mathcal{B}(H)$ of all bounded linear operators on H. It is well known that an operator A is Moore-Penrose invertible (as an element of $\mathcal{B}(H)$) if and only if it is *normally solvable*, that is, if and only if its range is closed. In particular, if $\dim H < \infty$, then every operator in $\mathcal{B}(H)$ is Moore-Penrose invertible. The following result is also well known.

Theorem 16.3. *Let $a \in PC$. The operator $T(a)$ is normally solvable if and only if it is Fredholm.*

Heinig and Hellinger [36] studied the following problem.

Question I. *Let $a \in PC$ and suppose $T(a)$ is normally solvable. Do the Moore-Penrose inverses $T_n^+(a)$ of $T_n(a)$ converge strongly to the Moore-Penrose inverse $T^+(a)$ of $T(a)$?*

The answer is as follows.

Theorem 16.4 (Heinig and Hellinger 1994). *Let $a \in PC$ and let $T(a)$ be normally solvable and thus Fredholm. Suppose also that $\mathrm{Ind}\,T(a) \neq 0$. Then $T_n^+(a)$ converges strongly to $T^+(a)$ in exactly two cases, namely,*

(a) *if $\mathrm{Ind}\,T(a) > 0$ and the Fourier coefficients $(a^{-1})_m$ of a^{-1} vanish for all sufficiently large m,*

(b) *if* $\operatorname{Ind} T(a) < 0$ *and the Fourier coefficients* $(a^{-1})_{-m}$ *of* a^{-1} *vanish for all* m *large enough.*

Clearly, (a) or (b) are satisfied in rare cases only. As the next result shows, for properly piecewise continuous functions, these conditions are even never met.

Corollary 16.5. *If* $a \in PC \setminus C$ *and* $T(a)$ *is normally solvable, then the following are equivalent:*

(i) $T_n^+(a) \to T^+(a)$ *strongly*;

(ii) $T(a)$ *is invertible,* $T_n(a)$ *is invertible for all sufficiently large* n*, and* $T_n^{-1}(a) \to T^{-1}(a)$ *strongly.*

Proof. The implication (ii)⇒(i) is trivial. To prove the reverse implication, assume (i) holds. If $T(a)$ is invertible, then (ii) follows from Corollary 10.2 and Proposition 1.1. So assume $T(a)$ is not invertible. Then, by Theorems 16.3 and 16.4, the function a^{-1} is a polynomial times a function in

$$\begin{aligned} H^\infty \cup \overline{H^\infty} \quad &= \big\{f \in L^\infty :\ f_n = 0 \ \text{ for } \ n < 0\big\} \\ &\cup \big\{f \in L^\infty :\ f_n = 0 \ \text{ for } \ n > 0\big\}. \end{aligned}$$

By a theorem of Lindelöf functions in $H^\infty \cup \overline{H^\infty}$ do not have jumps. As we supposed that $a \in PC \setminus C$, this completes the proof. ∎

Consequently, for symbols in $PC \setminus C$ Question I does not go beyond stability and the finite section method.

In algebraic language, Question I asks the following: if $T(a)$ is normally solvable, is $\{T_n(a)\}$ Moore-Penrose invertible in $\mathbf{S}(PC)$? Indeed, if $\{T_n(a)\}$ is Moore-Penrose invertible in $\mathbf{S}(PC)$, then there is a sequence $\{B_n\} \in \mathbf{S}(PC)$ such that

$$T_n(a)B_nT_n(a) - T_n(a) = 0, \quad B_nT_n(a)B_n - B_n = 0, \tag{61}$$

and

$$(T_n(a)B_n)^* - T_n(a)B_n = 0, \quad (B_nT_n(a))^* - B_nT_n(a) = 0 \qquad (62)$$

for all $n \geq 1$. From (61) and (62) we infer that $B_n = T_n^+(a)$, and passing to the strong limit $n \to \infty$ in (61) and (62), we get

$$\begin{aligned} &T(a)BT(a) - T(a) = 0, \quad BT(a)B - B = 0, \\ &(T(a))^* - T(a)B = 0, \quad (BT(a))^* - BT(a) = 0, \end{aligned}$$

which shows that the strong limit B of $T_n^+(a)$ coincides with $T^+(a)$. Conversely, suppose the answer to Question I is in the affirmative. Then (61) and (62) are satisfied for $B_n = T_n^+(a)$, and since $B_n \to T^+(a)$ strongly, the Banach-Steinhaus theorem implies that $\sup \|B_n\| < \infty$. Thus, $\{T_n(a)\}$ is Moore-Penrose invertible in the (big) C^*-algebra $\mathbf{S}$ introduced in Section 7. From Corollary 16.2 we deduce that $\{T_n(a)\}$ is even Moore-Penrose invertible in $\mathbf{S}(PC)$.

We know that invertibility in $\mathbf{S}(PC)/\mathbf{N}$ amounts to stability and thus to a more natural question than invertibility in $\mathbf{S}(PC)$. Question I is equivalent to Moore-Penrose invertibility in $\mathbf{S}(PC)$. This led Silbermann to asking what the Moore-Penrose invertibility of $\{T_n(a)\} + \mathbf{N}$ in $\mathbf{S}(PC)/\mathbf{N}$ means. Arguing as in the preceding paragraph, it is easily seen that this is just the following question.

Question II. *Let $a \in PC$ and suppose $T(a)$ is normally solvable. Is there a sequence $\{B_n\}$ of $n \times n$ matrices such that*

$$\|T_n(a)B_nT_n(a) - T_n(a)\| \to 0, \quad \|B_nT_n(a)B_n - B_n\| \to 0, \qquad (63)$$

$$\|(T_n(a)B_n)^* - T_n(a)B_n\| \to 0, \quad \|(B_nT_n(a))^* - B_nT_n(a)\| \to 0, \qquad (64)$$

and $B_n \to T^+(a)$ strongly?

Here is the answer to Question II.

Theorem 16.6 (Silbermann 1996). *If $a \in PC$ and $T(a)$ is normally solvable, then there is a sequence $\{B_n\} \in \mathbf{S}(PC)$ such that* (63) *and* (64) *hold and $B_n \to T^+(a)$ strongly.*

The proof of Theorem 16.6 is based on the following nice result, which is certainly of independent interest.

Theorem 16.7 (Roch and Silbermann 1996). *Let $a \in PC$. If $T(a)$ is Fredholm of index $k \in \mathbf{Z}$, then the $|k|$ first singular values of $T_n(a)$ go to zero, while the remaining $n - |k|$ singular values of $T_n(a)$ stay away from zero.*

An "elementary" proof of this theorem is in [19, Propositions 4.7 and 4.8].

Proof of Theorem 16.6. By Theorem 16.3, we may assume that $T(a)$ is Fredholm of some index $k \in \mathbf{Z}$. Let

$$(0 \le)\ s_1(T_n(a)) \le s_2(T_n(a)) \le \ldots \le s_n(T_n(a))$$

be the singular values of $T_n(a)$ and put

$$S_n := \operatorname{diag}\Big(s_1(T_n(a)), s_2(T_n(a)), \ldots, s_n(T_n(a))\Big).$$

There are unitary $n \times n$ matrices U_n and V_n such that

$$T_n(a) = U_n S_n V_n$$

(singular value decomposition). Let $S_n^{\#}$ be the diagonal matrix that arises from S_n by replacing the first $|k|$ singular values $s_1(T_n(a)), \ldots, s_{|k|}(T_n(a))$ with zero, and put

$$A_n^{\#} := U_n S_n^{\#} V_n.$$

Theorem 16.7 shows that

$$\|T_n(a) - A_n^{\#}\| = \|S_n - S_n^{\#}\| = s_{|k|}(T_n(a)) = o(1),$$

whence

$$\{T_n(a)\} + \mathbf{N} = \{A_n^{\#}\} + \mathbf{N}. \tag{65}$$

The eigenvalues of $(A_n^{\#})^* A_n^{\#} = V_n^* (S_n^{\#})^* S_n^{\#} V_n$ are the diagonal entries of the diagonal matrix $(S_n^{\#})^* S_n^{\#}$, that is, they are the numbers

$$\underbrace{0, \ldots, 0}_{|k|}, \Big(s_{|k|+1}(T_n(a))\Big)^2, \ldots, \Big(s_n(T_n(a))\Big)^2.$$

From Theorem 16.7 we therefore see that

$$\operatorname{sp}\Big(\{A_n^{\#}\}^* \{A_n^{\#}\}\Big) \subset \{0\} \cup [d^2, \infty)$$

for some $d > 0$, and Theorem 16.1 now shows that $\{A_n^\#\}$ is Moore-Penrose invertible in $\mathbf{S}(PC)$. It follows that $\{A_n^\#\} + \mathbf{N}$ is Moore-Penrose invertible in $\mathbf{S}(PC)/\mathbf{N}$, and (65) now implies that $\{T_n(a)\} + \mathbf{N}$ is Moore-Penrose invertible in $\mathbf{S}(PC)/\mathbf{N}$. ■

17. Quarter-plane Toeplitz operators

Let $Q = \{(i,j) \in \mathbf{Z}^2 : i \geq 1, j \geq 1\}$. For $a \in L^\infty(\mathbf{T}^2)$, the quarter-plane Toeplitz operator $T_{++}(a)$ is the bounded operator on $l^2(Q)$ which acts by the rule

$$(T_{++}(a)x)_{ij} = \sum_{\substack{k \geq 1 \\ l \geq 1}} a_{i-k,j-l} x_{kl},$$

where $\{a_{mn}\}_{m,n=-\infty}^{\infty}$ is the sequence of the Fourier coefficients of a,

$$a_{mn} = \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} a(e^{i\theta}, e^{i\varphi}) e^{-im\theta} e^{-in\varphi} \, d\theta \, d\varphi.$$

We define the projections $P_n \otimes P_n$ on $l^2(Q)$ by

$$\Big((P_n \otimes P_n)x\Big)_{ij} = \begin{cases} x_{ij} & \text{if } 1 \leq i \leq n \text{ and } 1 \leq j \leq n, \\ 0 & \text{otherwise,} \end{cases}$$

and we put

$$T_{n,n}(a) := (P_n \otimes P_n) \, T_{++}(a) \, (P_n \otimes P_n) | \mathrm{Im} \, (P_n \otimes P_n).$$

The sequence $\{T_{n,n}(a)\}_{n=1}^{\infty}$ is said to be stable if

$$\limsup_{n \to \infty} \|T_{n,n}^{-1}(a) \, (P_n \otimes P_n)\| < \infty.$$

The purpose of this section is to show how C^*-algebra techniques can be used to establish a stability criterion for $\{T_{n,n}(a)\}$ in the case where a is piecewise continuous.

The space $l^2(Q)$ is the Hilbert space tensor product of two copies of l^2,

$$l^2(Q) = l^2 \otimes l^2.$$

This means that finite sums $\sum_j x^{(j)} \otimes y^{(j)}$ are dense in $l^2(Q)$, where, for $x = \{x_k\} \in l^2$ and $y = \{y_k\} \in l^2$, the sequence $x \otimes y \in l^2(Q)$ is given by $x \otimes y = \{x_k y_l\}$.

Given $A, B \in \mathcal{B}(l^2)$, the operator $A \otimes B \in \mathcal{B}(l^2(Q))$ is defined as the linear and continuous extension to $l^2(Q)$ of the map defined for $x, y \in l^2$ by

$$(A \otimes B)(x \otimes y) := Ax \otimes By.$$

One can show that $\|A \otimes B\| = \|A\| \, \|B\|$. Also notice that if $A \in \mathcal{B}(l^2)$ and $\gamma \in \mathbf{C}$, then

$$\gamma I \otimes A = I \otimes \gamma A, \quad A \otimes \gamma I = \gamma A \otimes I.$$

For $b, c \in L^\infty$, we define $b \otimes c \in L^\infty(\mathbf{T}^2)$ by

$$(b \otimes c)(\xi, \eta) := b(\xi) c(\eta), \quad (\xi, \eta) \in \mathbf{T}^2.$$

We let $PC \otimes PC$ stand for the closure in $L^\infty(\mathbf{T}^2)$ of the set of all finite sums

$$\sum_j b_j \otimes c_j, \quad b_j \in PC, \ c_j \in PC.$$

If $a \in PC \otimes PC$ is a finite sum $a = \sum_j b_j \otimes c_j$ with $b_j, c_j \in PC$, then

$$T_{++}(a) = \sum_j T(b_j) \otimes T(c_j), \quad T_{n,n}(a) = \sum_j T_n(b_j) \otimes T_n(c_j).$$

Our aim is to prove the following result.

Theorem 17.1. *Let $a \in PC \otimes PC$. The sequence $\{T_{n,n}(a)\}$ is stable if and only if the four operators*

$$T_{++}(a_{00}), \quad T_{++}(a_{01}), \quad T_{++}(a_{10}), \quad T_{++}(a_{11}) \tag{66}$$

are invertible, where for $(\xi, \eta) \in \mathbf{T}^2$,

$$a_{00}(\xi, \eta) = a(\xi, \eta), \quad a_{01}(\xi, \eta) = a(\xi, \eta^{-1}),$$
$$a_{10}(\xi, \eta) = a(\xi^{-1}, \eta), \quad a_{11}(\xi, \eta) = a(\xi^{-1}, \eta^{-1}).$$

Invertibility criteria for general quarter-plane Toeplitz operators are not known. For symbols in $PC \otimes PC$, we have at least an effectively verifiable Fredholm criterion. It is easily seen that for every $a \in PC \otimes PC$ and each $(\xi, \eta) \in \mathbf{T}^2$ the four limits

$$a(\xi \pm 0, \eta \pm 0) := \lim_{\substack{\varepsilon \to 0 \pm 0, \\ \delta \to 0 \pm 0}} a(\xi e^{i\varepsilon}, \eta e^{i\delta}).$$

exist. Thus, for $a \in PC \otimes PC$ each $(\tau, \mu) \in \mathbf{T} \times [0,1]$ we can define two functions $a^1_{\tau,\mu}$ and $a^2_{\tau,\mu}$ in PC by

$$\begin{aligned} a^1_{\tau,\mu}(t) &= (1-\mu)\, a\,(\tau - 0, t) + \mu a\,(\tau + 0, t), \\ a^2_{\tau,\mu}(t) &= (1-\mu)\, a\,(t, \tau - 0) + \mu a\,(t, \tau + 0), \quad t \in \mathbf{T}. \end{aligned} \tag{67}$$

Here is a Fredholm criterion.

Theorem 17.2 (Duduchava 1977). *Let $a \in PC \otimes PC$. The operator $T_{++}(a)$ is Fredholm on $l^2(Q)$ if and only if the operators $T(a^1_{\tau,\mu})$ and $T(a^2_{\tau,\mu})$ are invertible for all $(\tau, \mu) \in \mathbf{T} \times [0,1]$. In that case* $\operatorname{Ind} T_{++}(a) = 0$.

We now proceed to the proof of Theorem 17.1. Let $\mathbf{F}$ stand for the C^*-algebra of all sequences $\{A_n\}_{n=1}^\infty$ of operators

$$A_n \in \mathcal{B}\Big(\operatorname{Im}(P_n \otimes P_n)\Big)$$

such that

$$\|\{A_n\}\| := \sup_{n \geq 1} \|A_n\| < \infty,$$

and let $\mathbf{M}$ be the closed ideal of $\mathbf{F}$ which consists of all sequence $\{A_n\}$ for which $\|A_n\| \to 0$ as $n \to \infty$. Let $\mathbf{S}$ and $\mathbf{N}$ be as in Section 7. Given two C^*-subalgebras $\mathbf{E}$ and $\mathbf{G}$ of $\mathbf{S}/\mathbf{N}$, we define the C^*-subalgebra $\mathbf{E} \otimes \mathbf{G}$ of $\mathbf{F}/\mathbf{M}$ as the closure in $\mathbf{F}/\mathbf{M}$ of the set of all finite sums of the form

$$\sum_j \Big\{E_n^{(j)} \otimes G_n^{(j)}\Big\} + \mathbf{M}, \quad \{E_n^{(j)}\} + \mathbf{N} \in \mathbf{E},\ \{G_n^{(j)}\} + \mathbf{N} \in \mathbf{G}.$$

Put

$$\mathbf{A} := \mathbf{S}(PC)/\mathbf{N}.$$

It is easily seen that if $a \in PC \otimes PC$, then the sequence $\{T_{n,n}(a)\}$ is stable if and only if $\{T_{n,n}(a)\} + \mathbf{M}$ is invertible in $\mathbf{A} \otimes \mathbf{A}$.

In what follows, we use the abbreviations

$$\{A_n\}^\nu := \{A_n\} + \mathbf{N}, \quad \{A_n\}^\mu := \{A_n\} + \mathbf{M}.$$

Lemma 17.3. *Let $a \in PC \otimes PC$. Then $\{T_{n,n}(a)\}^\mu$ is invertible in $\mathbf{A} \otimes \mathbf{A}$ if and only if the four operators* (66) *as well as the two* *elements*

$$\{T_{n,n}(a)\}^\mu + \mathbf{J} \otimes \mathbf{A} \in \mathbf{A} \otimes \mathbf{A}/\mathbf{J} \otimes \mathbf{A}, \tag{68}$$
$$\{T_{n,n}(a)\}^\mu + \mathbf{A} \otimes \mathbf{J} \in \mathbf{A} \otimes \mathbf{A}/\mathbf{A} \otimes \mathbf{J} \tag{69}$$

are invertible.

Proof. The invertibility of the elements (68) and (69) is equivalent to the invertibility of the element

$$\{T_{n,n}(a)\}^\mu + \mathbf{J} \otimes \mathbf{J} \in \mathbf{A} \otimes \mathbf{A}/\mathbf{J} \otimes \mathbf{J}. \tag{70}$$

Indeed, the invertibility of (68) and (69) is obviously implied by the invertibility of (70). Conversely, if (68) and (69) have the inverses

$$\{B_n\}^\mu + \mathbf{J} \otimes \mathbf{A} \ \text{ and } \ \{C_n\}^\mu + \mathbf{A} \otimes \mathbf{J},$$

then

$$\Big\{B_n + C_n - B_n T_{n,n}(a)\, C_n\Big\}^\mu + \mathbf{J} \otimes \mathbf{J}$$

is readily seen to be the inverse of (70).

Let $\mathbf{J}_0$ and $\mathbf{J}_1$ be the ideals introduced in Section 8. The ideal $\mathbf{J} \otimes \mathbf{J}$ is the smallest closed ideal of $\mathbf{A} \otimes \mathbf{A}$ which contains the four ideals

$$\mathbf{J}_0 \otimes \mathbf{J}_0, \quad \mathbf{J}_0 \otimes \mathbf{J}_1, \quad \mathbf{J}_1 \otimes \mathbf{J}_0, \quad \mathbf{J}_1 \otimes \mathbf{J}_1.$$

The maps

$$\begin{aligned}
\psi_{00}&: \{A_n\}^\mu \mapsto \operatorname*{s-lim}_{n\to\infty} (P_n \otimes P_n) A_n (P_n \otimes P_n),\\
\psi_{01}&: \{A_n\}^\mu \mapsto \operatorname*{s-lim}_{n\to\infty} (P_n \otimes W_n) A_n (P_n \otimes W_n),\\
\psi_{10}&: \{A_n\}^\mu \mapsto \operatorname*{s-lim}_{n\to\infty} (W_n \otimes P_n) A_n (W_n \otimes P_n),\\
\psi_{11}&: \{A_n\}^\mu \mapsto \operatorname*{s-lim}_{n\to\infty} (W_n \otimes W_n) A_n (W_n \otimes W_n),
\end{aligned}$$

are well-defined $*$-homomorphisms from $\mathbf{A} \otimes \mathbf{A}$ to $\mathcal{B}(l^2(Q))$, and the restrictions

$$\psi_{00}|\mathbf{J}_0 \otimes \mathbf{J}_0,\ \psi_{01}|\mathbf{J}_0 \otimes \mathbf{J}_1,\ \psi_{10}|\mathbf{J}_1 \otimes \mathbf{J}_0,\ \psi_{11}|\mathbf{J}_1 \otimes \mathbf{J}_1$$

are injective. Let us, for instance, prove the injectivity of $\psi_{01}|\mathbf{J}_0 \otimes \mathbf{J}_1$. Finite sums of the form

$$\sum_j \{P_n K_j P_n \otimes W_n L_j W_n\}^\mu, \quad K_j \in \mathcal{K},\ L_j \in \mathcal{K}.$$

are dense in $\mathbf{J}_0 \otimes \mathbf{J}_1$. Since

$$\begin{aligned}
&\Big\|\sum_j K_j \otimes L_j\Big\| \\
&\quad = \lim_{n\to\infty}\Big\|(P_n \otimes W_n)\sum_j (P_n K_j P_n \otimes W_n L_j W_n)(P_n \otimes W_n)\Big\| \\
&\quad \le \limsup_{n\to\infty}\Big\|\sum_j (P_n K_j P_n \otimes W_n L_j W_n)\Big\| \\
&\quad = \Big\|\sum_j \{P_n K_j P_n \otimes W_n L_j W_n\}^\mu\Big\|,
\end{aligned}$$

we see that every element of $\mathbf{J}_0 \otimes \mathbf{J}_1$ is of the form

$$\{A_n\}^\mu = \Big\{(P_n \otimes W_n) A (P_n \otimes W_n)\Big\}^\mu \tag{71}$$

with some $A \in \mathcal{K}(l^2(Q))$. But if $\{A_n\}^\mu$ is given by (71), then $\psi_{01}(\{A_n\}^\mu) = A$, and hence $\psi_{01}(\{A_n\}^\mu) = 0$ implies that $\{A_n\}^\mu = 0$.

Thus, Theorem 5.6 shows that $\{T_{n,n}(a)\}^\mu$ is invertible if and only if (70) (equivalently: (68) and (69)) and the four operators

$$T_{++}(a_{\varepsilon\delta}) = \psi_{\varepsilon\delta}\Big(\{T_{n,n}(a)\}^\mu\Big), \quad (\varepsilon, \delta) \in \{0,1\} \times \{0,1\}$$

are invertible. ■

Thus, to prove Theorem 17.1 we are left with showing that (68) and (69) are automatically invertible if the four operators (66) are invertible.

Let $\mathbf{A} \otimes \mathbf{C}$ be the closure in $\mathbf{A} \otimes \mathbf{A}$ of the set of all finite sums

$$\sum_j \{B_n \otimes \gamma P_n\}^\mu, \quad \{B_n\}^\nu \in \mathbf{A},\ \gamma \in \mathbf{C}.$$

Since $B_n \otimes \gamma P_n = \gamma B_n \otimes P_n$, it follows that $\mathbf{A} \otimes \mathbf{C}$ coincides with the closure in $\mathbf{A} \otimes \mathbf{A}$ of the set of all finite sums

$$\sum_j \{B_n \otimes P_n\}^\mu, \quad \{B_n\}^\nu \in \mathbf{A}.$$

Clearly $\mathbf{A}\otimes\mathbf{C}$ is a C^*-subalgebra of $\mathbf{A}\otimes\mathbf{A}$. By virtue of Proposition 5.3(b), the image of $\mathbf{A}\otimes\mathbf{C}$ under the canonical $*$-homomorphism

$$\mathbf{A}\otimes\mathbf{A}\to\mathbf{A}\otimes\mathbf{A}/\mathbf{J}\otimes\mathbf{A}$$

is a C^*-subalgebra of $\mathbf{A}\otimes\mathbf{A}/\mathbf{J}\otimes\mathbf{A}$. We denote this C^*-subalgebra by $\mathbf{A}\otimes\mathbf{C}/\mathbf{J}\otimes\mathbf{A}$.

Lemma 17.4. *The map*

$$\mathbf{A}/\mathbf{J}\to\mathbf{A}\otimes\mathbf{C}/\mathbf{J}\otimes\mathbf{A},\quad \{B_n\}^\nu+\mathbf{J}\mapsto\{B_n\otimes P_n\}^\mu+\mathbf{J}\otimes\mathbf{A}\tag{72}$$

is a well-defined $*$-isomorphism.

Proof. The map

$$\mathbf{A}\to\mathbf{A}\otimes\mathbf{C}/\mathbf{J}\otimes\mathbf{A},\quad \{B_n\}^\nu\mapsto\{B_n\otimes P_n\}^\mu+\mathbf{J}\otimes\mathbf{A}.\tag{73}$$

is a surjective $*$-homomorphism, and we must show that its kernel is $\mathbf{J}$. So suppose $\{B_n\}^\nu\in\mathbf{A}$ and

$$\{B_n\otimes P_n\}^\mu\in\mathbf{J}\otimes\mathbf{A}\tag{74}$$

Sequences in $\mathbf{J}$ and $\mathbf{A}$ have strong limits in $\mathcal{K}$ and $\mathcal{A}(PC)$, respectively. Thus, on denoting by $B\in\mathcal{A}(PC)$ the strong limit of B_n, we infer from (74) that

$$B\otimes I\in\mathcal{K}\otimes\mathcal{A}(PC).$$

This implies that

$$\|B-P_nB\|=\|(B-P_nB)\otimes I\|=\|B\otimes I-(P_n\otimes I)(B\otimes I)\|=o(1)$$

as $n\to\infty$, and as P_nB has finite rank, it follows that $B\in\mathcal{K}$. Consequently, the Gelfand transform $\Gamma(B+\mathcal{K})$ vanishes identically. From Theorems 6.4 and 8.5 we therefore deduce that $\Gamma(\{B_n\}^\nu+\mathbf{J})$ is also identically zero, and Theorem 5.4 so shows that $\{B_n\}^\nu\in\mathbf{J}$. ■

Lemma 17.5. *The C^*-algebra $\mathbf{A}\otimes\mathbf{C}/\mathbf{J}\otimes\mathbf{A}$ is a C^*-subalgebra of the center of the C^*-algebra $\mathbf{A}\otimes\mathbf{A}/\mathbf{J}\otimes\mathbf{A}$.*

Proof. For $\{B_n\},\{C_n\},\{D_n\}\in\mathbf{A}$ we have

$$\{B_n\otimes P_n\}^\mu\{C_n\otimes D_n\}^\mu=\{B_nC_n\otimes D_n\}^\mu,$$
$$\{C_n\otimes D_n\}^\mu\{B_n\otimes P_n\}^\mu=\{C_nB_n\otimes D_n\}^\mu,$$

and since $\{B_nC_n - C_nB_n\}^\nu \in \mathbf{J}$ due to the commutativity of $\mathbf{A}/\mathbf{J}$ (Theorem 8.5), we arrive at the assertion. ■

Lemma 17.5 enables us to have recourse to the local principle of Allan and Douglas (Theorem 5.5). By virtue of Lemma 17.4 and Theorem 8.5, we can identify the maximal ideal space of $\mathbf{A}\otimes\mathbf{C}/\mathbf{J}\otimes\mathbf{A}$ with $\mathbf{T}\times[0,1]$. In accordance with Theorem 5.5, we associate with $(\tau,\mu)\in\mathbf{T}\times[0,1]$ the smallest closed ideal $J_{\tau,\mu}$ of $\mathbf{A}\otimes\mathbf{A}/\mathbf{J}\otimes\mathbf{A}$ which contains the set

$$\Big\{\{B_n\otimes P_n\}^\mu + \mathbf{J}\otimes\mathbf{A}:\ \Big(\Gamma\Big(\{B_n\}^\nu+\mathbf{J}\Big)\Big)(\tau,\mu)=0\Big\}.$$

Lemma 17.6. *Let $a\in PC\otimes PC$, $(\tau,\mu)\in\mathbf{T}\times[0,1]$, and define $a^1_{\tau,\mu}\in PC$ by (67). Then*

$$\Big(\{T_{n,n}(a)\}^\mu+\mathbf{J}\otimes\mathbf{A}\Big)+J_{\tau,\mu}=\Big(\Big\{P_n\otimes T_n(a^1_{\tau,\mu})\Big\}^\mu+\mathbf{J}\otimes\mathbf{A}\Big)+J_{\tau,\mu}. \tag{75}$$

Proof. Since

$$\Big\|\Big\{T_{n,n}(a)\Big\}^\mu+\mathbf{J}\otimes\mathbf{A}\Big\|\le\limsup_{n\to\infty}\|T_{n,n}(a)\|\le\|a\|_\infty$$

and

$$\begin{aligned}&\Big\|\Big\{P_n\otimes T_n(a^1_{\tau,\mu})\Big\}^\mu+\mathbf{J}\otimes\mathbf{A}\Big\|\le\limsup_{n\to\infty}\Big\|P_n\otimes T_n(a^1_{\tau,\mu})\Big\|\\&\le\|a^1_{\tau,\mu}\|_\infty=\sup_{t\in\mathbf{T}}|(1-\mu)a(\tau-0,t)+\mu a(\tau+0,t)|\\&\le(1-\mu)\|a\|_\infty+\mu\|a\|_\infty=\|a\|_\infty,\end{aligned}$$

it suffices to prove (75) in the case where a is a finite sum

$$a=\sum_j b_j\otimes c_j,\quad b_j\in PC,\ c_j\in PC.$$

In that case we have, by Theorem 8.5,

$$\begin{aligned}a^1_{\tau,\mu}&=(1-\mu)\sum_j b_j(\tau-0)c_j+\mu\sum_j b_j(\tau+0)c_j,\\&=\sum_j\Big(\Gamma\Big(\{T_n(b_j)\}^\nu+\mathbf{J}\Big)\Big)(\tau,\mu)c_j\\&=:\sum_j\gamma_j(\tau,\mu)c_j,\end{aligned}$$

whence

$$\begin{aligned}
&\Big\{T_{n,n}(a)\Big\}^{\mu} - \Big\{P_n \otimes T_n(a^1_{\tau,\mu})\Big\}^{\mu} + \mathbf{J} \otimes \mathbf{A} \\
&= \sum_j \Big\{T_n(b_j) \otimes T_n(c_j)\Big\}^{\mu} - \sum_j \Big\{P_n \otimes \gamma_j(\tau,\mu) T_n(c_j)\Big\}^{\mu} + \mathbf{J} \otimes \mathbf{A} \\
&= \sum_j \Big\{T_n(b_j) \otimes T_n(c_j)\Big\}^{\mu} - \sum_j \Big\{\gamma_j(\tau,\mu) P_n \otimes T_n(c_j)\Big\}^{\mu} + \mathbf{J} \otimes \mathbf{A} \\
&= \sum_j \Big\{\Big(T_n(b_j) - \gamma_j(\tau,\mu) P_n\Big) \otimes T_n(c_j)\Big\}^{\mu} + \mathbf{J} \otimes \mathbf{A}. \qquad (76)
\end{aligned}$$

Because

$$\begin{aligned}
&\bigg(\Gamma\Big(\Big\{T_n(b_j) - \gamma_j(\tau,\mu) P_n\Big\}^{\nu} + \mathbf{J}\Big)\bigg)(\tau,\mu) \\
&= \bigg(\Gamma\Big(\Big\{T_n(b_j)\Big\}^{\nu} + \mathbf{J}\Big)\bigg)(\tau,\mu) - \gamma_j(\tau,\mu) = 0,
\end{aligned}$$

we see from the definition of $J_{\tau,\mu}$ that (76) belongs to $J_{\tau,\mu}$. ■

Proof of Theorem 17.1. As already said, it remains to prove that (68) and (69) are invertible if the four operators (66) are invertible. The sole Fredholmness of the operator $T_{++}(a_{00}) = T_{++}(a)$ implies that $T(a^1_{\tau,\mu})$ is invertible for all $(\tau,\mu) \in \mathbf{T} \times [0,1]$ (Theorem 17.2). Hence, $\{T_n(a^1_{\tau,\mu})\}^{\nu}$ is invertible in $\mathbf{A}$ (Theorem 9.2). Let $\{B_n\}^{\nu}$ be the inverse of $\{T_n(a^1_{\tau,\mu}\}^{\nu}$. Then $\{P_n \otimes B_n\}^{\mu}$ is the inverse of $\{P_n \otimes T_n(a^1_{\tau,\mu})\}^{\mu}$, that is

$$\Big\{P_n \otimes T_n(a^1_{\tau,\mu})\Big\}^{\mu}$$

is invertible in $\mathbf{A} \otimes \mathbf{A}$ for all $(\tau,\mu) \in \mathbf{T} \times [0,1]$. From Lemma 17.6 we now deduce that

$$\Big(\{T_{n,n}(a)\}^{\mu} + \mathbf{J} \otimes \mathbf{A}\Big) + J_{\tau,\mu}$$

is invertible for all $(\tau,\mu) \in \mathbf{T} \times [0,1]$, and Theorem 5.5 therefore gives the invertibility of (68). Replacing in the above reasoning $\mathbf{A} \otimes \mathbf{C}/\mathbf{J} \otimes \mathbf{A}$ by $\mathbf{C} \otimes \mathbf{A}/\mathbf{A} \otimes \mathbf{J}$, one can analogously prove that (69) is invertible. ■

18. Segal-Bargmann space Toeplitz operators

The Segal-Bargmann space $A^2(\mathbf{C}^N, d\mu)$ is the Hilbert space of all functions f which are analytic on $\mathbf{C}^N$ and satisfy

$$\|f\|^2 := \int_{\mathbf{C}^N} |f(z_1, \ldots, z_N)|^2 \, d\mu(z_1) \ldots d\mu(z_N) < \infty,$$

where $d\mu(z) = (2\pi)^{-1} e^{-|z|^2} dA(z)$ and $dA(z)$ is area measure on $\mathbf{C}$.

Let us first consider the case $N = 1$. We denote by $\overline{\mathbf{C}}$ the compactification of $\mathbf{C}$ by a circle at infinity. Thus, $a \in C(\overline{\mathbf{C}})$ if and only if the limit

$$a_\infty(t) = \lim_{r \to \infty} a(rt)$$

exists for every $t \in \mathbf{T}$ and if $a_\infty \in C(\mathbf{T})$. For $a \in C(\overline{\mathbf{C}})$, the Toeplitz operator $T^1(a)$ on $A^2(\mathbf{C}, d\mu)$ is defined by

$$(T^1(a)f)(z) = \int_{\mathbf{C}} a(w) e^{z\overline{w}/2} f(w) \, d\mu(w), \quad z \in \mathbf{C}.$$

An orthonormal basis $\{e_n\}_{n=1}^\infty$ of $A^2(\mathbf{C}, d\mu)$ is constituted by the functions

$$e_n(z) = \left(2^{n-1}(n-1)!\right)^{-1/2} z^{n-1}.$$

For $a \in C(\overline{\mathbf{C}})$, let $T^0(a)$ be the bounded operator on $A^2(\mathbf{C}, d\mu)$ whose matrix representation with respect to the basis $\{e_n\}_{n=1}^\infty$ is the Toeplitz matrix $T(a_\infty) = ((a_\infty)_{j-k})_{j,k=1}^\infty$ constituted by the Fourier coefficients of the function a_∞. One can show that

$$T^1(a) = T^0(a) + K \tag{77}$$

with some compact operator K on $A^2(\mathbf{C}, d\mu)$. We define the projections P_n on $A^2(\mathbf{C}, d\mu)$ by

$$P_n : \sum_{k=1}^\infty x_k e_k \mapsto \sum_{k=1}^n x_k e_k,$$

and we put $T_n^1(a) := P_n T^1(a) P_n | \operatorname{Im} P_n$. From Corollary 10.3 and (77) we immediately obtain the following.

Theorem 18.1. *Let $a \in C(\overline{\mathbf{C}})$. The sequence $\{T_n^1(a)\}_{n=1}^\infty$ is stable if and only if $T^1(a)$ and $T^0(a^0)$ are invertible, where $a^0(z) := a(\overline{z})$ for $z \in \mathbf{C}$.* ■

We now turn to the case $N = 2$. The space $A^2(\mathbf{C}^2, d\mu)$ is the Hilbert space tensor product of two copies of $A^2(\mathbf{C}, d\mu)$. We let $C(\overline{\mathbf{C}}^2)$ stand for the closure in $L^\infty(\mathbf{C}^2)$ of the set of all finite sums $\sum_j b_j \otimes c_j$ with $b_j, c_j \in C(\overline{\mathbf{C}})$. For a finite sum $a = \sum_j b_j \otimes c_j \in C(\overline{\mathbf{C}}^2)$, we define four operators $T^{\gamma,\delta}(a)$ ($\gamma, \delta \in \{0,1\}$) by

$$T^{\gamma,\delta}(a) = \sum_j T^\gamma(b_j) \otimes T^\delta(c_j). \tag{78}$$

One can show that $\|T^{\gamma,\delta}(a)\| \leq \|a\|_\infty$, which allows us to extend the definition of $T^{\gamma,\delta}(a)$ to all functions $a \in C(\overline{\mathbf{C}}^2)$. The really interesting operator is the operator $T^{1,1}(a)$, which can also be given by the formula

$$(T^{1,1}(a)f)(z) = \int_{\mathbf{C}^2} a(w) e^{z\overline{w}/2} f(w)\, d\mu(w), \quad z \in \mathbf{C}^2,$$

where $z\overline{w} = z_1\overline{w_1} + z_2\overline{w_2}$ and $d\mu(w) = d\mu(w_1)d\mu(w_2)$. The operator $T^{0,0}(a)$ is obviously unitarily equivalent to the operator $T_{++}(a)$ studied in Section 17. Put

$$T_n^{\gamma,\delta}(a) = (P_n \otimes P_n) T^{\gamma,\delta}(a)(P_n \otimes P_n) | \mathrm{Im}\,(P_n \otimes P_n).$$

Taking into account (78), it is easy to see that

$$\begin{aligned}
&(P_n \otimes P_n) T_n^{\gamma,\delta}(a)(P_n \otimes P_n) \to T^{\gamma,\delta}(a^{1,1}) \ \text{ strongly},\\
&(P_n \otimes W_n) T_n^{\gamma,\delta}(a)(P_n \otimes W_n) \to T^{\gamma,0}(a^{1,0}) \ \text{ strongly},\\
&(W_n \otimes P_n) T_n^{\gamma,\delta}(a)(W_n \otimes P_n) \to T^{0,\delta}(a^{0,1}) \ \text{ strongly},\\
&(W_n \otimes W_n) T_n^{\gamma,\delta}(a)(W_n \otimes W_n) \to T^{0,0}(a^{0,0}) \ \text{ strongly},
\end{aligned}$$

where

$$\begin{aligned}
a^{1,1}(z_1, z_2) &= a(z_1, z_2), \quad & a^{1,0}(z_1, z_2) &= a(z_1, \overline{z_2}),\\
a^{0,1}(z_1, z_2) &= a(\overline{z_1}, z_2), \quad & a^{0,0}(z_1, z_2) &= a(\overline{z_1}, \overline{z_2}).
\end{aligned}$$

Using arguments similar to those of Section 17, one can prove the following.

Theorem 18.2. *Let $a \in C(\overline{\mathbf{C}}^2)$ and let $(\gamma, \delta) \in \{0,1\}^2$. The sequence $\{T_n^{\gamma,\delta}(a)\}_{n=1}^\infty$ is stable if and only if the four operators*

$$T^{\gamma,\delta}(a^{1,1}), \quad T^{\gamma,0}(a^{1,0}), \quad T^{0,\delta}(a^{0,1}), \quad T^{0,0}(a^{0,0})$$

are invertible.

In the case $(\gamma, \delta) = (1,1)$ we encounter the four operators

$$T^{1,1}(a^{1,1}), \quad T^{1,0}(a^{1,0}), \quad T^{0,1}(a^{0,1}), \quad T^{0,0}(a^{0,0}),$$

that is, the stability of the sequence $\{T_n^{1,1}(a)\}$ of the truncations of a pure Segal-Bargmann space Toeplitz operator is determined by this operator itself, by the pure quarter-plane Toeplitz operator $T^{0,0}(a^{0,0})$, and by the two “mixed Toeplitz operators” $T^{1,0}(a^{1,0})$ and $T^{0,1}(a^{0,1})$.

For $(\gamma, \delta) = (0,0)$, Theorem 18.2 is the $C \otimes C$ (and thus a special) version of Theorem 17.1. Notice that the operators

$$T^{0,0}(a^{1,1}), \quad T^{0,0}(a^{1,0}), \quad T^{0,0}(a^{0,1}), \quad T^{0,0}(a^{0,0}),$$

are just the operators

$$T_{++}(a_{00}), \quad T_{++}(a_{01}), \quad T_{++}(a_{10}), \quad T_{++}(a_{11})$$

(in this order).

In the case of general N, we associate with every $a \in C(\overline{\mathbf{C}}^N)$ and every $\varepsilon = (\varepsilon_1, \dots, \varepsilon_N) \in \{0,1\}^N$ the operator

$$T^\varepsilon(a) = T^{\varepsilon_1,\dots,\varepsilon_N}(a) \in \mathcal{B}\Big(A^2(\mathbf{C}^N, d\mu)\Big)$$

and a function

$$a^\varepsilon = a^{\varepsilon_1,\dots,\varepsilon_N} \in C(\overline{\mathbf{C}}^N)$$

in the natural manner. We put

$$T_n^\varepsilon(a) = (P_n \otimes \dots \otimes P_n) T^\varepsilon(a) (P_n \otimes \dots \otimes P_n) | \mathrm{Im}\,(P_n \otimes \dots \otimes P_n).$$

Theorem 18.3. *Let $a \in C(\overline{\mathbf{C}}^N)$ and $\varepsilon \in \{0,1\}^N$. The sequence $\{T_n^\varepsilon(a)\}_{n=1}^\infty$ is stable if and only if the 2^N operators $T^{\varepsilon\delta}(a^\delta)$ ($\delta \in \{0,1\}^N$) are invertible; here*

$$\varepsilon \cdot \delta = (\varepsilon_1, \dots, \varepsilon_N) \cdot (\delta_1, \dots, \delta_N) := (\varepsilon_1 \delta_1, \dots, \varepsilon_N \delta_N).$$

The proof of Theorem 18.3 is rather difficult. It is done by induction on N and based on ideas analogous to the ones of Section 17. For pure Segal-Bargmann space Toeplitz operators, Theorem 18.3 reads as follows.

Theorem 18.4. *Let $a \in C(\overline{\mathbf{C}}^N)$. The sequence $\{T_n^{1,\ldots,1}(a)\}_{n=1}^{\infty}$ is stable if and only if the 2^N operators $T^\delta(a^\delta)$ $(\delta \in \{0,1\}^N)$ are invertible.*

It is certainly amusing to know that the proof of Theorem 18.3 is much simpler than the proof of Theorem 18.4. To be more precise: the only proof of Theorem 18.4 I have is first to prove Theorem 18.3 and then to deduce Theorem 18.4 as a special case from Theorem 18.4; I do not know how to prove Theorem 18.4 in a direct way.

The mystery's resolution is "the researcher's paradox": sometimes a proof by induction does only work if the result to be proved is sufficiently sharp. Here is an example of this experience taken from [60].

We want to prove that

$$\frac{1 \cdot 3 \cdot 5 \ldots (2N-1)}{2 \cdot 4 \cdot 6 \ldots 2N} < \frac{1}{\sqrt{N}}. \tag{79}$$

Since $1/2 < 1$, this is true for $N = 1$. Suppose (79) holds for N. To prove (79) for $N + 1$, we consider

$$\frac{1 \cdot 3 \cdot 5 \ldots (2N+1)}{2 \cdot 4 \cdot 6 \ldots (2N+2)}, \tag{80}$$

and (79) for N tells us that (80) is at most

$$\frac{1}{\sqrt{N}} \frac{2N+1}{2N+2}.$$

But the inequality

$$\frac{1}{\sqrt{N}} \frac{2N+1}{2N+2} < \frac{1}{\sqrt{N+1}}$$

is equivalent to the inequality

$$4N^3 + 8N^2 + 5N + 1 < 4N^3 + 8N^2 + 4N,$$

which is definitely not true.

Now let us replace (79) by the stronger assertion

$$\frac{1 \cdot 3 \cdot 5 \ldots (2N-1)}{2 \cdot 4 \cdot 6 \ldots 2N} < \frac{1}{\sqrt{N+1}}. \tag{81}$$

As $1/2 < 1/\sqrt{2}$, this is valid for $N = 1$. Assuming that (81) is true for N, we get

$$\frac{1 \cdot 3 \cdot 5 \ldots (2N+1)}{2 \cdot 4 \cdot 6 \ldots (2N+2)} < \frac{1}{\sqrt{N+1}} \frac{2N+1}{2N+2},$$

and because the inequality

$$\frac{1}{\sqrt{N+1}} \frac{2N+1}{2N+2} < \frac{1}{\sqrt{N+2}}$$

is equivalent to the inequality

$$4N^3 + 12N^2 + 9N + 2 < 4N^3 + 12N^2 + 12N + 4,$$

we get indeed the desired result

$$\frac{1 \cdot 3 \cdot 5 \ldots (2N+1)}{2 \cdot 4 \cdot 6 \ldots (2N+2)} < \frac{1}{\sqrt{N+2}}.$$

19. Notes and some history

The first treatise on the numerical analysis of Toeplitz matrices is Gohberg and Feldman's book [30]. This book has strongly influenced the subsequent development, and it already contains such results as Corollary 10.2.

In the sixties, I.B. Simonenko created his local principle for studying the Fredholmness of so-called operators of local type. Inspired by the success of this theory, V.B. Dybin proposed to apply local principles to the investigation of the stability of approximation methods. A.V. Kozak, a student of Dybin's, carried out this program; he generalized Simonenko's local principle to general Banach algebras and used this new tool to dispose of the question of the stability of sequences of compressions of multidimensional convolution operators with continuous symbols [38], [39]. For example, Theorem 17.1 for continuous symbols is a 1973 result of Kozak. Due to the work of Gohberg, Feldman, and Kozak, to mention only the principal figures, the stability theory for one- and higher-dimensional Toeplitz operators (on l^2 or on l^p) with continuous symbols was essentially complete in the middle of the seventies.

As the Fredholm theory of Toeplitz operators with piecewise continuous symbols is especially beautiful for l^p spaces, people started investigating the stability of $\{T_n(a)\}$ for $a \in PC$ on l^p. In 1977, Verbitsky and Krupnik [62] were able to prove that if $a \in PC$ has only one jump, then for $\{T_n(a)\}$ to be stable on l^p it is necessary and sufficient that $T(a)$ and $T(\widetilde{a})$ be invertible on l^p. Clearly, this result cried for a local principle, but all attempts to prove it for symbols with at least two jumps failed.

In the late seventies, I became a student of Silbermann's and joined his research into the asymptotic behavior of Toeplitz determinants (Szegö limit theorems). We then read Widom's paper [64], which contained formula (31):

$$T_n(a)T_n(b) = T_n(ab) - P_nH(a)H(\widetilde{b})P_n - W_nH(\widetilde{a})H(b)W_n.$$

We soon understood how this formula can be used to study the stability of $\{T_n(a)\}$ on l^p for $a \in PC$ and extended the Verbitsky/Krupnik result to symbols with a finite number of jumps [16]. I was satisfied, but Silbermann went further. When tackling symbols with countably many jumps, he realized that local principles work perfectly modulo the ideal

$$\mathbf{J} = \Big\{\{P_nKP_n + W_nLW_n\} + \mathbf{N} : \ K, L \in \mathcal{K}\Big\}$$

(note that already the idea of checking whether this is at all an ideal is daring). In this way, Silbermann was able to prove results like the l^p version of Corollary 10.3, and at the same time he laid the foundation for a new level of the application of Banach algebra techniques to numerical analysis and thus for an approach that has led to plenty of impressive results during the last 18 years.

Another problem people were working on at that time was quarter-plane Toeplitz operators with symbols in $PC \otimes PC$. For symbols in $C \otimes C$, the Fredholm theory was settled earlier by I.B. Simonenko, R.G. Douglas, R. Howe, and V.S. Pilidi. In his 1977 paper [26], Duduchava localized over the central subalgebra

$$\mathcal{A}(C) \otimes \mathbf{C}/\mathcal{K} \otimes \mathcal{A}(PC) \cong C(\mathbf{T})$$

of $\mathcal{A}(PC) \otimes \mathcal{A}(PC)/\mathcal{K} \otimes \mathcal{A}(PC)$, arrived at local representatives of the form

$$T(b) \otimes T(c) + I \otimes T(d),$$

and he was able to come up with these rather complicated local representatives and to establish Theorem 17.2. It was again the attempt of extending an l^2 result to the l^p setting that moved things ahead: I understood that localization over the larger central subalgebra

$$\mathcal{A}(PC) \otimes \mathbf{C}/\mathcal{K} \otimes \mathcal{A}(PC) \cong C(\mathbf{T} \times [0,1])$$

of $\mathcal{A}(PC) \otimes \mathcal{A}(PC)/\mathcal{K} \otimes \mathcal{A}(PC)$ results in local representatives of the pretty nice form

$$I \otimes T(d),$$

which simplifies Duduchava's l^2 theory significantly and, moreover, also yields the l^p version of Theorem 17.2 [7].

Together with Silbermann, we then realized that a localization analogous to the one sketched in the previous paragraph should also work when studying the stability of $\{T_{n,n}(a)\}$ for $a \in PC \otimes PC$. The result was our 1983 paper [17], in which we introduced the C^*-algebra

$$\mathbf{A} = \mathbf{S}(PC)/\mathbf{N},$$

established Theorem 8.5 for $\mathbf{A}/\mathbf{J}$, localized over

$$\mathbf{A} \otimes \mathbf{C}/\mathbf{J} \otimes \mathbf{A} \cong C(\mathbf{T} \times [0,1]),$$

obtained local representatives of the form

$$\Big\{P_n \otimes T_n(d)\Big\},$$

and arrived at Theorem 17.1. As to the best of my knowledge, this was the first time C^*-algebras were deliberately used in connection with a question of numerical analysis.

Subsequently C^*-algebras have been applied to many problems of numerical analysis, and things have now become a big business. The general strategy is always to translate the numerical problem into a question about invertibility in some Banach or C^*-algebra and then to find ideals so that the quotient algebras have a sufficiently large center. Then one can employ a local principle (say Theorem 5.5), and in the end one has to "lift" things back from the quotient algebras to the original algebra. (e.g. by Theorem 5.6). The price one has to pay is that one must pass from individual operators to algebras of operators, and these usually contain objects that are much more complicated than the original operator.

Working in C^*-algebras is always more comfortable than being in Banach algebras, and results obtained with the help of C^*-algebra techniques are often much sharper than those gained by having recourse to Banach algebras. Moreover, frequently C^*-algebras automatically do a perfect job for us. For instance, Gohberg's result

$$a \in PC, \;\; T(a) \text{ invertible } \Longrightarrow \limsup_{n\to\infty} \|T_n^{-1}(a)\| < \infty$$

is 30 years old – only 5 years ago it was observed that the simple C^*-algebra argument

$$\text{preservation of spectra } \Longrightarrow \text{ preservation of norms}$$

yields almost at once the undreamt-of improvement

$$a \in PC \Longrightarrow \lim_{n\to\infty} \|T_n^{-1}(a)\| = \|T^{-1}(a)\|.$$

The material of this text exhibits only a modest piece of the business and is restricted to the few things I have participated in and to my favorite operators, the Toeplitz operators. For more about the topic the reader is referred to my texts [19] and [11] with Silbermann and Grudsky and to Hagen, Roch, and Silbermann's inexhaustible books [33], [34]. An independent line of the development has its root in Arveson's works [3], [4]. The ideas of Arveson are also discussed in [34].

In what follows I give a few sources for the result of the text.

Sections 1 to 3. Standard.

Section 4. The books [30], [25], [45], [18], [19] contain the results of this section, historical comments, and much more. For Hankel operators, we also refer to the books [46], [47].

Section 5. Most of the results of this section are well known. Theorem 5.5 is due to Allan [1], who established even the Banach algebra version of this theorem. The theorem was independently discovered by Douglas [25], who was the first to understand the relevancy of such a theorem in operator theory. A special version of Theorem 5.6 is already in Silbermann's paper [54]. In the form we cited Theorem 5.6, it appears in Roch and Silbermann's works [49], [51].

Section 6. Theorem 6.2 is due to Gohberg [28] and Coburn [23], Proposition 6.3 and Theorem 6.4 were established by Gohberg and Krupnik [31].

Sections 7 to 10. Corollary 10.2 is a 1967 result of Gohberg [29], Corollary 10.3 was obtained in Silbermann's 1981 paper [54], and Theorem 10.1 is from my 1983 paper [17] with Silbermann. Theorems 8.1 and 9.1 are C^*-algebra modifications of similar results by Silbermann [54]. The C^*-algebra $(\mathbf{S}/(PC)/\mathbf{N})/\mathbf{J}$ was introduced in [17]. In that paper we also proved Theorems 8.3 to 8.5 and Theorem 9.2. The two proofs of Theorem 9.2 given here are from Hagen, Roch, and Silbermann's book [34]. They use heavy guns, but they are beautiful concrete illustrations of the C^*-algebra machinery. More "elementary" proofs are in [17], [18, Theorem 7.11], [19, Theorem 3.11].

Section 11. The results and the C^*-algebra approach of this section are from my article [9], which, I must admit, was essentially influenced by Silbermann's paper [55]. It should also be noted that the goal of [9] was to give alternative proofs and extensions of the results by Reichel and Trefethen [48] on the pseudospectra of Toeplitz matrices (see Section 15).

For sequences in $\mathbf{S}(C)$, that is, for continuous symbols, we now also have proofs of Theorems 11.1 and 11.2 that do not invoke C^*-algebras [15]. The techniques developed in this connection allow passage to operators on l^p and, perhaps more importantly, they yield good two-sided estimates for the differences

$$\|A_n\| - \max(\|A\|, \|\widetilde{A}\|), \quad \|A_n^{-1}\| - \max(\|A^{-1}\|, \|\widetilde{A}^{-1}\|)$$

and, in particular, estimates for the speed with which the norms $\|T_n^{-1}(a)\|$ converges to their limit $\|T^{-1}(a)\|$; see [32], [11], [14], [13], [12].

Sections 12 and 13. Corollary 12.3 is even true for arbitrary real-valued $a \in L^\infty$ and is a classical result: Szegö [58] and Widom [63]. Corollary 13.2 is due to Widom [65] and Silbermann [55]. Theorems 12.2 and 13.1 were established by Roch and Silbermann [50].

Section 14. The classical paper by Schmidt and Spitzer is [57]. An alternative proof of Theorem 14.1, which also gives the "limiting measure", is in Hirschman's work [37]. Notice that both proofs are nevertheless rather intricate and that a "transparent" proof of Theorem 14.1 is still missing.

Section 15. Henry Landau [40], [41] was the first to study ε-pseudospectra of *Toeplitz matrices*, and Corollary 15.3 (for smooth symbols a) is in principle already in his papers. Independently, Corollary 15.3 (for symbols with absolutely convergent Fourier series) was discovered by Reichel and Trefethen [48]. These three authors derived the result with the help of different methods. The approach of Section 15 and Corollary 15.3 for symbols in PC is from my paper [9]. Theorem 15.2 appeared in the paper [51] by Roch and Silbermann for the first time.

For matrices (= operators on $\mathbf{C}^n$), Theorem 15.1 is a simple fact. In the form stated here it was probably first proved by T. Finck and T. Ehrhardt (see [51]).

I also recommend

`http://web.comlab.ox.ac.uk/projects/pseudospectra`

which is a wonderful webpage by Mark Embree and Nick Trefethen.

Section 16. For symbols with absolutely convergent Fourier series, Theorem 16.4 was proved by Heinig and Hellinger [36] using different methods. For symbols in PC, the theorem was established by Silbermann [56]. A very simple proof was subsequently given in my paper [10]; this proof is also in [19, Section 4.10].

Several versions of Questions I and II have been studied for a long time; see, for example, the paper [43] by Moore and Nashed.

In the context of Toeplitz operators, Question II was first raised by Silbermann [56], who also obtained Theorem 16.6. The "splitting property" described by Theorem 16.7 was discovered by Roch and Silbermann [52]; their proof makes essential use of C^*-algebras (and even gives an analogue of Theorem 16.7 for block Toeplitz matrices). The "elementary" proof alluded to in the text was found in [10] (and it does not work for block Toeplitz matrices).

Section 17. Theorem 17.2 was established by Duduchava [26] (a simpler proof and the extension to l^p are in [7]). All other results and the reasoning of this section are from the paper [17] by Silbermann and myself.

Section 18. These results were obtained in my paper [8] and my papers [20], [21], [22] with Wolf.

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