A GROWTH THEOREM FOR BIHOLOMORPHIC MAPPINGS ON A BANACH SPACE

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Abstract. Let $\|\cdot\|$ be an arbitrary norm on a Banach space E. Let B be the open unit ball of E for the norm $\|\cdot\|$, and let $f : B \to E$ be a biholomorphic convex mapping such that $f(0) = 0$ and $df(0)$ is identity. We will give an upper bound of the growth of f.

1. Introduction

Let $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ denote the open unit disc in the complex plane $\mathbb{C}$. Let $f : \Delta \to \mathbb{C}$ be a biholomorphic convex mapping with $f(0) = 0$ and $f'(0) = 1$. Then the following inequality holds :

$$|f(z)| \leq \frac{|z|}{1 - |z|}, \text{ for all } z \in \Delta.$$

It is natural to consider a generalization of the growth theorem above to $\mathbb{C}^n$. Let Ω be a domain in $\mathbb{C}^n$ which contains the origin in $\mathbb{C}^n$. A holomorphic mapping $f : \Omega \to \mathbb{C}^n$ is said to be normalized, if $f(0) = 0$ and the Jacobian matrix $Df(0)$ at the origin is identity. Let $\mathbb{B}^n$ denote the Euclidean unit ball in $\mathbb{C}^n$. Let $f : \mathbb{B}^n \to \mathbb{C}^n$ be a normalized biholomorphic convex mapping. Then C. H. FitzGerald and C. R. Thomas [4], T. Liu [11] and T. J. Suffridge [12] extended the upper bound above for the growth of f to $\mathbb{B}^n$ in $\mathbb{C}^n$ by using different methods and showed that

$$\|f(z)\|_2 \leq \frac{\|z\|_2}{1 - \|z\|_2} \text{ for all } z \in \mathbb{B}^n,$$

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where $\|\cdot\|_2$ denotes the Euclidean norm. Let

$$B_p = \{z = (z_1, \dots, z_n) \in \mathbb{C}^n : \|z\|_p = (\sum_{i=1}^{n} |z_i|^p)^{1/p} < 1\}$$

for $p \geq 1$ and

$$D(p_1, \dots, p_n) = \{z = (z_1, \dots, z_n) \in \mathbb{C}^n : |z_1|^{p_1} + \cdots + |z_n|^{p_n} < 1\}$$

with $p_1, \dots, p_n \geq 1$. S. Gong and T. Liu [7] gave the upper bound above for the growth of normalized biholomorphic convex mappings on B_p and $D(p_1, \dots, p_n)$ and H. Hamada [8] proved a similar result on the unit ball in $\mathbb{C}^n$ with respect to an arbitrary norm. In this paper, we give the upper bound for the growth of a biholomorphic convex mapping on the unit ball in a complex Banach space as follows.

Main Theorem *Let E be a complex Banach space with norm $\|\cdot\|$. Let B be the open unit ball of E for the norm $\|\cdot\|$, and let $f : B \to E$ be a biholomorphic convex mapping such that $f(0) = 0$ and $df(0)$ is identity. Then*

$$\|f(z)\| \leq \frac{\|z\|}{1 - \|z\|}$$

for all $z \in B$.

2. Notation and preliminaries

Let U be an open set in a complex normed space E and let F be a complex Banach space. Let f be a holomorphic mapping from U to F. Then the following equation holds in a neighbourhood V of x in U for $x \in U$:

$$f(z) = \sum_{n=1}^{\infty} P_n(z - x), \tag{2.1}$$

where

$$P_n(y) = \frac{d^n f(x)}{n!}(y) = \frac{1}{2\pi\sqrt{-1}} \int_{|\zeta|=1} \frac{1}{\zeta^{n+1}} f(x + \zeta y) d\zeta$$

for any $y \in E \setminus \{0\}$ such that $x + \zeta y \in U$ for all $\zeta \in \mathbb{C}$ with $|\zeta| \leq 1$. The series (2.1) is called the Taylor expansion of f by n-homogeneous polynomials P_n at x.

Let E be a complex Banach space with norm $\|\cdot\|$. Let B be the open unit ball of E for the norm $\|\cdot\|$, and let $f : B \to E$ be a biholomorphic mapping. A biholomorphic mapping f is said to be convex if $f(B)$ is convex.

The following theorem (the Maximum Modulus Principle) is well-known (see, for example, Dunford and Schwartz [3]).

Theorem 2.1 *Let E be a complex Banach space with norm $\|\cdot\|$. Let Δ be the unit disc in $\mathbb{C}$, and let $f : \Delta \to E$ be a holomorphic mapping. If there exists a point $\zeta_0 \in \Delta$ such that $\|f(\zeta_0)\| = \sup\{\|f(\zeta)\| : \zeta \in \Delta\}$, then $\|f(\zeta)\|$ is constant on Δ.*

3. Proof of Main Theorem

Let Δ be the unit disc in $\mathbb{C}$. We take a fixed boundary point $w \in \partial B$. Let $f(z) = \sum_{n=1}^{\infty} P_n(z)$ be the expansion of f by n-homogeneous polynomials P_n in a neighbourhood V of 0 in E. Then we have $f(z) = z + \sum_{n=2}^{\infty} P_n(z)$. For $\zeta \in \Delta$,

$$f(\zeta w) = \zeta w + \sum_{n=2}^{\infty} \zeta^n P_n(w). \tag{3.1}$$

Let $m \geq 2$, $m \in \mathbb{Z}$ be fixed. Let $a = \exp(2\pi\sqrt{-1}/m)$. Then

$$\begin{aligned}\sum_{k=0}^{m-1} f(\zeta^{\frac{1}{m}} a^k w) &= \sum_{k=0}^{m-1} \{\zeta^{\frac{1}{m}} a^k w + \sum_{n=2}^{\infty} (\zeta^{\frac{1}{m}} a^k)^n P_n(w)\} \\ &= \zeta^{\frac{1}{m}} (\sum_{k=0}^{m-1} a^k) w + \sum_{n=2}^{\infty} (\sum_{k=0}^{m-1} a^{kn}) \zeta^{\frac{n}{m}} P_n(w) \\ &= m \sum_{j=1}^{\infty} \zeta^j P_{jm}(w).\end{aligned}$$

This is holomorphic with respect to $\zeta \in \Delta$. Since $f(B)$ is convex, we have

$$\frac{1}{m} \sum_{k=0}^{m-1} f(\zeta^{\frac{1}{m}} a^k w) \in f(B).$$

We set

$$h(\zeta) = f^{-1}(\frac{1}{m}\sum_{k=0}^{m-1} f(\zeta^{\frac{1}{m}} a^k w)).$$

Then $h(\zeta)$ is holomorphic on Δ. By the conditions on f,

$$f^{-1}(z) = z + O(\|z\|^2).$$

We have

$$\begin{aligned} h(\zeta) &= f^{-1}(\sum_{j=1}^{\infty} \zeta^j P_{jm}(w)) \\ &= \sum_{j=1}^{\infty} \zeta^j P_{jm}(w) + O(\|\sum_{j=1}^{\infty} \zeta^j P_{jm}(w)\|^2) \\ &= \zeta P_m(w) + O(|\zeta|^2). \end{aligned}$$

Therefore $\frac{h(\zeta)}{\zeta}$ is a holomorphic mapping from Δ into E. If $\varepsilon > 0$ is sufficiently small, $\frac{h(\zeta)}{\zeta}$ is continuous and holomorphic on the set $\{\zeta : |\zeta| \leq 1 - \varepsilon\}$. Since $h(\Delta) \subset B$, by Theorem 2.1, we obtain

$$\left\|\frac{h(\zeta)}{\zeta}\right\| < \frac{1}{1-\varepsilon}$$

on $\{\zeta : |\zeta| \leq 1 - \varepsilon\}$. Letting ε tend to 0, we have

$$\|P_m(w)\| = \left\| \left.\frac{h(\zeta)}{\zeta}\right|_{\zeta=0} \right\| \leq 1$$

for all $m \geq 2$. Then we have

$$\begin{aligned}\|f(\zeta w)\| &\leq \|\zeta w\| + \|\sum_{n=2}^{\infty} \zeta^n P_n(w)\| \\ &\leq |\zeta| + \sum_{n=2}^{\infty} |\zeta|^n \\ &= \frac{|\zeta|}{1-|\zeta|} \\ &= \frac{\|\zeta w\|}{1-\|\zeta w\|}.\end{aligned}$$

and the proof of the Main Theorem is complete. ■

Let D be a bounded convex balanced domain in a complex Banach space E. The Minkowski function $N_D(z)$ of D is defined by

$$N_D(z) = \inf\{\alpha > 0 : z \in \alpha D\}$$

for all $z \in E$. Then $N_D(z)$ is a norm on E. By the Main Theorem, we obtain the following corollary.

Corollary *Let D be a bounded convex balanced domain in a complex Banach space E. let $f : D \to E$ be a biholomorphic convex mapping such that $f(0) = 0$ and $df(0)$ is identity. Then*

$$N_D(f(z)) \leq \frac{N_D(z)}{1 - N_D(z)}$$

for all $z \in D$.

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