Book Review

Complex Analysis on Infinite Dimensional Spaces

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One might attempt to define infinite dimensional holomorphy as the study of differentiable non-linear functions on infinite dimensional (usually complex) topological vector spaces. But then this immediately strikes us as so very remote from the growing field that it now is. The present picture of infinite dimensional holomorphy intersects with many branches of mathematics: differential geometry, Jordan algebras, Lie groups, operator theory, logic. It is an area of vigorous research, with connections to many other areas.

It is generally agreed that infinite dimensional holomorphy begins before the XXth century, and the works usually cited as the first in the field are often also cited as being the roots of Functional Analysis. Volterra uses the Taylor series expansion of a real-valued analytic function on $C[a, b]$ as early as 1887, von Koch and Hilbert use monomial expansions converging on polydiscs around the turn of the century. Fréchet (1909) defines and uses real polynomials on R^N and $C[a, b]$, and Gateaux (1912) defines complex holomorphic functions and proves a Cauchy integral formula and Cauchy inequalities. These first advances were on concrete spaces. Functional Analysis was in its infancy, with the definition of normed space still years away. The Polish school was interested in the non-linear: one can find still unsolved problems on infinite variable polynomials in The Scottish Book, mostly by Mazur and Orlicz, and Banach himself promised 'another volume' regarding analyticity would follow his Théorie des Opérations Linéaires. But this was not to be.

The second period of the development of infinite holomorphy starts in America in the 1930's, with Michal and his students. It is a period of unification and clarification of the theory. The relationship between k-homogeneous polynomial and k-linear mappings is now established, the polarization formula appears. Taylor series now converge on balanced domains of abstract spaces, and much of the finite-variable theory is generalized. In the 1940's and 1950's, tensor products make their appearance, as well as holomorphy on general locally convex spaces. But linear Functional Analysis overshadows the non-linear in this era, and progress is relatively slow, except in the gray area between the linear and the non-linear: the bilinear and its relation with duality and with the product in topological algebras.

The third era begins in the mid 1960's with Nachbin and his students. Infinite holomorphy comes of age, now it's what doesn't happen in finite dimensions that counts. Holomorphic functions of diverse types (integral, nuclear, weakly continuous) appear, spaces of holomorphic functions have competing topologies, τ_0, τ_ω, τ_δ, which may or may not coincide. Their duals are studied. Preduals are searched for, as are decompositions into spaces of homogeneous polynomials. Holomorphic automorphisms in infinite dimensions are studied, together with bounded symmetric domains and Jordan triple systems. The Levi problem, domains of holomorphy, plurisubharmonicity, germs of holomorphic functions all make their necessary appearance. Many open problems remain and are being actively studied today. In the mid 1980's polynomials and spaces of homogeneous polynomials began to call attention for their own sake and, fuelled by important advances in Banach space theory, now constitute a highly active field of study. This book appears then, at a time when it is very necessary, for not only has the subject grown, but its focus has changed since the 1980's when most infinite holomorphy texts were published.

Formally, the author assumes only basic knowledge of one complex variable and some experience with Banach space theory. The reader who approaches the book with that background alone will find many sections are hard but worthwhile reading. But the expert will also find much to learn from this book. As the author says in the preface, he has 'tried to maintain a delicate balance between ... self-contained introduction for the non-expert and to provide a comprehensive summary for the expert'. He has succeeded admirably in creating a book that admits several different readings; a book to come back to.

The unifying theme is the relationship in $H(U)$ (where U is an open subset of a locally convex space E) between the topologies τ_0, τ_ω and τ_δ; and the study of how this relationship is affected by properties of U and of the underlying locally convex space E. The theory of bounded symmetric domains and Jordan triples is not included.

The book begins with polynomials in both the multilinear and tensor product presentations. Topologies in spaces of polynomials are compared, and geometric properties of these spaces are discussed. Chapter 2 presents duality theory and reflexivity for spaces of polynomials. This is the opportunity to introduce several different classes of polynomials: approximable, nuclear, integral, weakly sequentially continuous, as well as the properties of the underlying space necessary to obtain positive results: the approximation property, nuclearity, the Radon-Nikodým property, non-containment of ℓ_1. Gateaux and Fréchet holomorphic mappings make their appearance in Chapter 3, together with the ways of approximating them, monomial and Taylor series expansions. The different topologies on spaces of holomorphic mappings are introduced and their properties studied. In Chapter 4, the approximation of holomorphic functions are applied to the comparison of topologies on $H(U)$ for a balanced open subset U of a Fréchet space. The comparison of topologies is then carried out in Chapter 5 for arbitrary open subsets U. This entails a discussion of Riemann domains over locally convex spaces, and of the Levi problem and Cartan-Thullen theorem. The book ends with the study of holomorphic extensions from dense subspaces and from closed subspaces, and with a discussion of the space of holomorphic functions of bounded type as an algebra, and the study of its spectrum.

The pace of the book is a sustained one, but the exposition is fresh and full of short excursions into other topics as the need for these arises. Thus one finds a nice discussion of the (BB)-property, a short introduction to the theory of locally convex spaces, and an excursion to the local theory of Banach spaces. Further on, the role played by quojections, the need for DFN spaces and nuclearity. An introduction to Schauder decompositions, a discussion of Riemann domains, another of Arens-regularity. These are the threads which tie the infinite holomorphy core of the book to the rest of Analysis, giving it its place in mathematics and ultimately, its beauty.

This is a scholarly work. The author has made a fruitful effort at simplifying and unifying techniques and providing new proofs. Each chapter is accompanied by many exercises — over 250 in all — most of which are commented in an appendix, and by extensive and carefully researched historical notes. The list of references has over 850 entries. A very complete book. And at a very reasonable price, too.

But perhaps the most valuable thing that the author has put into this book is his insight and his perspective. His views on why such a property will or will not play a significant role in the future, and how the subject is developing are sprinkled liberally throughout the text. This is a great attraction for the reader, who will come back for more, and in each reading, will find it.

We mathematicians take pride in the timelessness of our work: mathematics — more than any other human endeavour — is forever. I sometimes wonder, then, why mathematics books grow old so fast. A mathematical classic is any book older than its reader. In my bookcase this definition includes only a venerable handful: Landau, Hardy-Littlewood-Polya, Kelley, Banach, Zygmund. Dineen will be — I am sure — a classic.

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