THE 40TH INTERNATIONAL MATHEMATICAL OLYMPIAD, 1999

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The 1999 International Mathematical Olympiad was held in Bucharest, Romania, from 10 to 22 July. A total of 450 students representing 81 countries and territories took part. The competition, as always, consisted of two four and a half hour examinations, each examination consisting of three challenging mathematics problems. Each student competed as an individual and medals were awarded to the top performers. One of the Irish contestants, Raja Mukherji, was awarded a bronze medal for the second successive year.

In January of 1999, a number of secondary school students were invited to attend training sessions in one of the training centres around the country: UCC, UCD, UL and NUIM. The training sessions continued until the end of April, and on Saturday 8 May 1999 those students who were still involved in the training programme participated in the Irish Mathematical Olympiad (see below). The top six performers were chosen to represent Ireland in Bucharest. The members of the Irish team in 1999 were:

Raja Mukherji, Drimnagh Castle School, Dublin
Evelyn Hickey, Athlone Community College, Athlone, Co. Westmeath
Darren O Driscoll, Bandon Grammar School, Bandon, Co. Cork
David Conlon, St. Mel's College, Co. Longford
Micheál Griffin, St. Brendan's College, Killarney, Co. Kerry
Micheal J. Ryan, Charleville C.B.S., Charleville.

With the support of the University of Limerick, Gordon Lessells organized a training camp at UL from 29 June to 2 July. As well as the team members, several others of the top performers in the Irish Mathematical Olympiad who will be eligible to compete for a place on the team next year were also invited to participate in the training camp.

The Competition

I was the Team Leader and was accompanied by Pat McCarthy of NUIM as observer. We arrived in Romania on Saturday, 10 July and were met at the airport by the competition organizers and brought to Poiana Brasov, high in the Carpathian mountains. The team leaders from all of the participating countries met there to choose the six problems for the competition from a shortlist of twenty seven problems selected by the Problem Committee of the host country (Romania). Each participating country had been invited earlier in the year to submit confidentially to the Problem Selection Committee a handful of potential problems–which must be entirely original and of an appropriate degree of difficulty–for the competition. It was from this long list of submitted problems that the twenty seven shortlisted problems had been chosen by the Problem Selection committee. The final problem on the shortlist was one of the problems submitted by Ireland (composed by Finbarr Holland and T. J. Laffey). The jury (i.e. the team leaders) judged it to be a very beautiful problem, but of too high a level of difficulty for the Olympiad. One of the jury members said that he would like to use it in a competition in his own country for graduate students of mathematics.

As soon as they began work, the jury realized that the set of shortlisted problems was of a higher than average level of difficulty and that the resulting final selection of six would almost inevitably lead to a very difficult competition, which proved to be the case. The selection of problems by the jury proceeded very efficiently and the final six problems were chosen and voted on by the afternoon of Monday, 12 July. The following morning was then given over to the translation of the problems from the official English version into all the languages of the competition. On Tuesday afternoon the translations were presented to and ratified by the jury, thus completing this initial stage of their work.

In the meantime, the Irish team, accompanied by the deputy leader, Gordon Lessells, arrived in Bucharest on Tuesday, 13 July. One of the team members, Raja Mukherji, had just arrived home (in Dublin) the previous day from Bangkok, Thailand, where he had been awarded a silver medal in the International Chemistry Olympiad as part of very successful Irish participation in this event. Indeed, it may have been the strain of travel and prolonged effort which led to his falling ill shortly after his arrival in Bucharest. He was admitted to hospital on Wednesday and, fortunately, released again the following morning in good health. The other team members, meanwhile, were shown some of the sights of Bucharest, including the Village Museum and the Botanic Gardens.

The opening ceremony took place in the Royal Palace Hall in Bucharest on Thursday afternoon, 15 July. The leaders, deputy leaders, participants and organizers were entertained by performances of music and dance ranging from traditional Romanian to modern and classical.

The exams took place in the Polytechnic University on Friday and Saturday 16 and 17 July. Both mornings, the leaders were taken to the Polytechnic University to answer queries from the students about the papers during the first half hour of the exams. The answer books of the students were delivered to the leaders at the Hotel Bucuresti after the exams both evenings. For the leaders, deputy leaders and observers, it was at this point that the long task of studying, and deciphering, their own team's efforts with the objective of obtaining as many points as possible began. Each of the six problems is assigned a team of 'coordinators' from the host country, and it is the job of the team leader, with the help of the deputy leader (and observer, if applicable), to present the efforts of each of their team members to the coordinators and to make the case, if a case exists, for one or more points. Preparation for these coordinating sessions can involve quite a lot of work. Indeed, Pat McCarthy stayed up all night on the first evening, poring over the efforts of our team from the first day's competition.

The Results

The difficulty of the IMO 1999 competition was reflected in the scores, which were much lower than average. In most years several contestants will achieve the maximum possible score of 42 points (each of the six problems is worth 7 points), but this year the highest score achieved–by three contestants–was 39 points. According to the IMO rules, the top twelfth of the contestants are awarded gold medals, the next sixth are awarded silver medals and the next one quarter are awarded bronze medals. The cutoff scores for these three categories of contestant were as follows: contestants scoring at least 28 points were awarded a gold medal, those whose score was in the range 19 to 27 were awarded a silver medal, and those whose score was in the range 12 to 18 obtained a bronze medal.

Considering the difficulty of the contest, which tested even the more intensively trained teams from the top performing countries, the Irish team's performance was more than satisfactory. The scores of the team members were as follows:

Raja Mukherji	12
David Conlon	9
M. J. Ryan	7
Darren O Driscoll	4
Evelyn Hickey	3
Micheál Griffin	3

Raja was awarded a bronze medal, having achieved the cutoff score of 12 points. This, along with all the other medals, was awarded at the closing ceremony, which took place at the enormous and opulent Palace of the Parliament (formerly Ceaucescu's palace) in Bucharest. The President of Romania, Emil Constantinescu, was present. He made a speech praising the contestants and organizers and personally presented some of the top prizes.

Here are the two papers of the Irish Mathematical Olympiad 1999, which were used to select this year's team, followed by the papers of the International Mathematical Olympiad. The time allowed for each paper of the Irish Mathematical Olympiad was three hours. The time allowed for each of the IMO papers was four and a half hours.

Solutions to the problems of the 40th IMO can be found at the back of the Bulletin.

Twelfth Irish Mathematical Olympiad

Saturday, 8 May 1999

First Paper

1. Find all real values of x which satisfy

$$\frac{x^2}{(x+1-\sqrt{x+1})^2} < \frac{x^2+3x+18}{(x+1)^2}.$$

2. Show that there is a positive number in the Fibonacci sequence which is divisible by 1000.
[The Fibonacci sequence F_n is defined by the conditions $F_0 = 0$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. So the sequence begins 0, 1, 1, 2, 3, 5, 8, 13,...]

3. Let D, E, F be points on the sides BC, CA, AB, respectively, of triangle ABC such that AD is perpendicular to BC, BE is the angle-bisector of $\angle B$ and F is the midpoint of AB. Prove that AD, BE, CF are concurrent if and only if

$$a^2(a-c) = (b^2-c^2)(a+c),$$

where a, b, c are the lengths of the sides BC, CA, AB, respectively, of triangle ABC.

4. A square floor consisting of 10000 squares (100 squares $\times$ 100 squares–like a large chessboard) is to be tiled. The only available tiles are rectangular 1×3 tiles, fitting exactly over three squares of the floor.

(i) If a 2×2 square is removed from the centre of the floor, prove that the remaining part of the floor can be tiled with the available tiles.
(ii) If, instead, a 2×2 square is removed from the corner, prove that the remaining part of the floor cannot be tiled with the available tiles.
[There are sufficiently many tiles available. To *tile* the floor–or a portion thereof–means to cover it completely with the tiles, each tile covering three squares, and no pair of tiles overlapping. The tiles may not be broken or cut.]

5. Three numbers $a < b < c$ are said to be in arithmetic progression if $c - b = b - a$.
Define a sequence u_n, $n = 0, 1, 2, 3, \ldots$, as follows: $u_0 = 0$, $u_1 = 1$ and for each $n \geq 1$, u_{n+1} is the smallest positive integer such that $u_{n+1} > u_n$ and $\{u_0, u_1, \ldots, u_n, u_{n+1}\}$ contains no three elements which are in arithmetic progression.
Find u_{100}.

Second Paper

6. Solve the system of (simultaneous) equations

$$y^2 = (x+8)(x^2+2),$$

$$y^2 - (8+4x)y + (16+16x-5x^2) = 0.$$

7. A function $f : \mathbb{N} \to \mathbb{N}$ (where $\mathbb{N}$ denotes the set of positive integers) satisfies
(1) $f(ab) = f(a)f(b)$ whenever the greatest common divisor of a and b is 1,
(2) $f(p+q) = f(p) + f(q)$ for all prime numbers p and q.
Prove that $f(2) = 2$, $f(3) = 3$ and $f(1999) = 1999$.

8. Let a, b, c and d be positive real numbers whose sum is 1. Prove that

$$\frac{a^2}{a+b} + \frac{b^2}{b+c} + \frac{c^2}{c+d} + \frac{d^2}{d+a} \geq \frac{1}{2},$$

with equality if and only if $a = b = c = d = 1/4$.

9. Find all positive integers m with the property that the fourth power of the number of (positive) divisors of m equals m.

10. $ABCDEF$ is a convex (not necessarily regular) hexagon with $AB = BC$, $CD = DE$, $EF = FA$ and

$$\angle ABC + \angle CDE + \angle EFA = 360°.$$

Prove that the perpendiculars from A, C and E to FB, BD and DF respectively are concurrent.

Fortieth International Mathematical Olympiad

First Day–16 July 1999

1. Determine all finite sets S of at least three points in the plane which satisfy the following condition:

for any two distinct points A and B in S, the perpendicular bisector of the line segment AB is an axis of symmetry for S.

2. Let n be a fixed integer, with $n \geq 2$.

(a) Determine the least constant C such that the inequality

$$\sum_{1\leq i<j\leq n} x_i x_j (x_i^2 + x_j^2) \leq C \left(\sum_{1\leq i\leq n} x_i \right)^4$$

holds for all real numbers $x_1, \ldots, x_n \geq 0$.

(b) For this constant C, determine when equality holds.

(Editor's note: in connection with this problem, see Finbarr Holland's article on pp.73–78 of this Bulletin.)

3. Consider an $n \times n$ square board, where n is a fixed even positive integer. The board is divided into n^2 unit squares. We say that two different squares on the board are *adjacent* if they have a common side. N unit squares on the board are marked in such a way that every square (marked or unmarked) on the board is adjacent to at least one marked square.

Determine the smallest possible value of N.

Second Day–17 July 1999

4. Determine all pairs (n, p) of positive integers such that p is a prime, $n \leq 2p$, and

$$(p-1)^{n+1} \text{ is divisible by } n^{p-1}.$$

5. Two circles Γ_1 and Γ_2 are contained inside a circle Γ, and are tangent to Γ at the distinct points M and N, respectively. Γ_1 passes through the centre of Γ_2. The line passing through the two points of intersection of Γ_1 and Γ_2 meets Γ at A and B. The lines MA and MB meet Γ_1 at C and D respectively. Prove that CD is tangent to Γ_2.

6. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x - f(y)) = f(f(y)) + xf(y) + f(x) - 1$$

for all x and y in $\mathbb{R}$.

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