SUMMING A DIVERGENT SERIES

D. G. C. McKeon

It is well known that the manipulation of divergent series can give peculiar results. The series $S(a) = \sum_{n=0}^{\infty} a^n n!$ is clearly horribly divergent for $a \neq 0$, yet we will show by using some elementary integrals involving the Bessel function $K_0(x)$ that it can be viewed as an expansion in powers of a of a function which diverges for $a < 0$.

The two integrals we need are [1]

$$\int_0^\infty dt\, t^n\, K_0(2\sqrt{t}) = \frac{1}{2}(n!)^2 \tag{1}$$

and

$$f(a) = 2\int_0^\infty dt\, e^{-at}\, K_0(2\sqrt{t}) = \frac{e^{\frac{1}{a}}}{a} \int_{1/a}^\infty dt\, \frac{e^{-t}}{t}\ (a > 0). \tag{2}$$

Let us consider expanding e^{-at} in the argument of (2) so that

$$f(a) = 2\int_0^\infty dt \sum_{n=0}^\infty \frac{(-a)^n}{n!}\, t^n K_0(2\sqrt{t}); \tag{3}$$

integration of this term by term using (1) yields

$$\begin{aligned} f(a) &= 2\sum_{n=0}^\infty \frac{(-a)^n}{n!} \left[\frac{1}{2}(n!)^2\right] \\ &= \sum_{n=0}^\infty (-a)^n n! \end{aligned} \tag{4}$$

Thus the divergent series $S(-a)$ can be considered as a representation of the function $f(a)$, which from its definition in (2), is clearly singular for $a < 0$! (We note that $f(a) \to 1$ as $a \to 0^+$ in both (2) and (4).) This may alternatively be viewed as a way of inserting a "convergence" factor of $1/(n!)^2$ into each term of the series for $S(-a)$.

These manipulations are akin to the insertion of a factor of $\frac{1}{n!}$ in Borel series. To see this, consider the integrals

$$\int_0^\infty dt\, t^n e^{-t} = n! \tag{5}$$

and

$$g(a) = \int_0^\infty dt\, e^{-at} e^{-t} = \frac{1}{1+a} \;(a > -1). \tag{6}$$

Expanding e^{-at} in powers of at in (6) we obtain

$$g(a) = \int_0^\infty dt \sum_{n=0}^{\infty} \frac{(-at)^n}{n!}\, e^{-t} \tag{7}$$

which becomes, if we integrate term-by-term using (5)

$$\begin{aligned} g(a) &= \sum_{n=0}^{\infty} \frac{(-a)^n}{n!}\,(n!) \qquad (8) \\ &= \sum_{n=0}^{\infty} (-a)^n. \end{aligned}$$

By comparing (6) with (8), we see that $\dfrac{1}{1+a}$ is formally represented by the series $\sum_{n=0}^{\infty} (-a)^n$ for all $a > -1$ even though this geometric series diverges for $|a| > 1$.

We hence see that, once again, a divergent series which at first glance is "meaningless" may in fact be given a "meaning", provided we are willing to indulge in interchanging the order of summation and integration, as we have done in going from (2) to (4) or (6) to (8). (As $\sum_{n=0}^{\infty} \frac{(-at)^n}{n!}$ uniformly converges to e^{-at} for finite at, term-by-term integration of the series in (3) and (7) is justified over any finite range.)

A more formal approach that is clearly related to the usual discussion of Borel Summation is now sketched. If one has a series

$$f(a) = \sum_{n=0}^{\infty} b_n a^n \, n! \tag{9}$$

which converges for $|a| < R$, then

$$\phi(a) = \sum_{n=0}^{\infty} \frac{b_n a^n}{n!} \tag{10}$$

converges everywhere. It is easily shown now, using the integral of eq. (1), that

$$F(a) = 2 \int_0^\infty dt \, \phi(ta) K_0(2\sqrt{t}) \tag{11}$$

is an analytic continuation of $f(a)$ across $|a| = R$ wherever $F(a)$ is analytic. We have essentially considered an example of this general result for the case $b_n = (-1)^n$ and $R = 0$ where the usual arguments are invalid.

References

[1] I. Gradsteyn and M. Ryzhik, Table of Integrals, Series and Products, 5th ed. Academic Press.

D. C. G. McKeon
Department of Applied Mathematics
University of Western Ontario
London
Canada N6A 5B7