IRREDUCIBLE CHARACTERS OF SMALL DEGREE OF THE UNITRIANGULAR GROUP

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We let q be a power of some prime p and for all $n \in \mathbb{N}$, we let $U_n(q)$ be the set of $n \times n$ matrices over $\mathbf{F}_q$ which are lower-triangular and have every diagonal entry equal to 1. $U_n(q)$ is called the *unitriangular group* of degree n over $\mathbf{F}_q$. $U_n(q)$ is a Sylow p–subgroup of $\mathrm{GL}_n(q)$, the general linear group of invertible $n \times n$ matrices over $\mathbf{F}_q$. For any finite group G, we will write $\mathrm{Irr}(G)$ for the set of irreducible complex characters of G. Isaacs [2] has shown that every element of $\mathrm{Irr}\big(U_n(q)\big)$ has degree a power of q. Using this result of Isaacs, a theorem of Huppert [1] for the case where $q = p$ can be extended to prove that

$$\Big\{\Gamma(1)\ :\ \Gamma \in \mathrm{Irr}\big(U_n(q)\big)\Big\} = \left\{q^a\ :\ 0 \le a \le \frac{n(n-1)}{2}\right\}.$$

In [4], the author enumerated the irreducible complex characters of $U_{2m}(q)$ having each of the three highest degrees, as well as the irreducible complex characters of $U_{2m-1}(q)$ of highest degree. In this paper, we consider the elements of $\mathrm{Irr}\big(U_n(q)\big)$ having each of the three lowest degrees, namely 1, q, and q^2. As the underlying field will remain fixed throughout, we will write U_n in place of $U_n(q)$.

It is easy to see that the commutator subgroup U_n' of U_n is given by

$$U_n' = \{(a_{ij}) \in U_n : a_{i,i-1} = 0 \text{ for } 2 \le i \le n\}.$$

We let $V_{n,1} = U_n/U_n'$. Then the elements of $\mathrm{Irr}(U_n)$ of degree 1 are obtained by extending the irreducible characters of $V_{n,1}$ to U_n.

As $V_{n,1}$ is an (elementary) abelian group of order q^{n-1}, this means that there are precisely q^{n-1} irreducible characters of U_n of degree 1. We will show that the irreducible characters of U_n of degrees q and q^2 may be obtained by extending characters of analogous factor groups and we will count the number having each degree.

For $0 \leq t \leq n-1$, we define

$$W_{n,t} = \{(a_{ij}) \in U_n : a_{ij} = 0 \text{ if } 1 < i - j \leq t\}.$$

Thus, for example, $W_{n,0} = U_n$, $W_{n,1} = U_n'$, $W_{n,3} = U_n''$, and $W_{n,n-1} = I$. Each $W_{n,t}$ is a normal subgroup of U_n. For $0 \leq t \leq n-1$ we define

$$V_{n,t} = U_n / W_{n,t}.$$

Thus the elements of $V_{n,t}$ may be identified with lower-triangular $n \times n$ matrices over $\mathbf{F}_q$ with 1's along the diagonal and for which we only consider the entries on the main diagonal and on the t diagonals below the main diagonal. We will write $\tilde{A}$ for the elements $AW_{n,t}$ of $V_{n,t}$.

For $1 \leq s \leq n$, we define

$$\begin{aligned} N_{n,s} &= \{(a_{ij}) \in U_n : a_{ij} = 0 \text{ for } i > j \text{ if either } i \leq s \text{ or } j > s\} \\ &= \left\{ \begin{pmatrix} I_s & 0 \\ C & I_{n-s} \end{pmatrix} : C \in M_{n-s,s}(\mathbf{F}_q) \right\} \end{aligned}$$

and

$$\begin{aligned} H_{n,s} &= \{(a_{ij}) \in U_n : a_{ij} = 0 \text{ for } i > j \text{ where } i > s \text{ and } j \leq s\} \\ &= \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} : A \in U_s,\, B \in U_{n-s} \right\}. \end{aligned}$$

For all s, $N_{n,s}$ is a normal abelian subgroup of U_n and $H_{n,s}$ is a subgroup of U_n which complements $N_{n,s}$, i.e. $N_{n,s} \cap H_{n,s} = I$ and $U = N_{n,s} H_{n,s}$. Usually we write the elements of $H_{n,s}$ as (A, B) instead of $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$.

We will make extensive use of the following lemma. It follows partly from a theorem of Gallagher, while the rest can be deduced from standard Clifford Theory. (See Corollary 6.17 and Problem 6.18 of [3].) The proof is straightforward and we omit it.

Lemma 1. *We let G be a finite group, N a normal abelian subgroup of G, and H a subgroup of G which complements N. We let λ be an element of $\mathrm{Irr}(N)$ and we let $T_G(\lambda)$ be the stabilizer of λ in G, i.e.*

$$T_G(\lambda) = \{g \in G : \lambda^g = \lambda\},$$

where $\lambda^g \in \mathrm{Irr}(N)$, $\lambda^g(n) = \lambda(g^{-1}ng)$ for all $n \in N$. We let $S_G(\lambda) = T_G(\lambda) \cap H \cong T_G(\lambda)/N$. Note that $T_G(\lambda) = S_G(\lambda)N$. We define a character $\bar{\lambda}$ of $T_G(\lambda)$ by $\bar{\lambda}(hn) = \lambda(n)$ for all h in $S_G(\lambda)$ and all n in N. Then for each irreducible character Ψ of $S_G(\lambda)$, $\left(\bar{\lambda}\Psi\right)^G$ is an element of $\mathrm{Irr}(G)$, distinct for distinct Ψ's, and

$$\lambda^G = \sum_{\Psi \in \mathrm{Irr}(S_G(\lambda))} \Psi(1) \left(\bar{\lambda}\Psi\right)^G.$$

Starting instead with an irreducible character Γ of G, if Γ lies over $\lambda \in \mathrm{Irr}(N)$, then there exists an irreducible character Ψ of $S_G(\lambda)$ such that λ occurs with multiplicity $\Psi(1)$ in $\Gamma|_N$ and

$$\Gamma(1) = \Psi(1) \frac{|H|}{|S_G(\lambda)|}. \quad \blacksquare$$

For $1 \leq s \leq n$ and for every $s \times (n-s)$ matrix D over $\mathbf{F}_q$, we define $\Lambda_D : N_{n,s} \to \mathbb{C}$ by

$$\Lambda_D\left(\begin{pmatrix} I_s & 0 \\ C & I_{n-s} \end{pmatrix}\right) = \omega^{T(\mathrm{tr}\,(CD))},$$

where ω is a primitive p^{th} root of unity in $\mathbb{C}$, $T : \mathbf{F}_q \to \mathbf{F}_p$ is the usual trace mapping from an extension field into the ground field, tr denotes the trace of a square matrix, and we identify the elements of $\mathbf{F}_p$ with the integers $0, 1, \ldots, p-1$.

Lemma 2. *For* $1 \leq s \leq n$, $\mathrm{Irr}(N_{n,s}) = \{\Lambda_D : D \in M_{s,n-s}\}$ *and* $\Lambda_{D_1} = \Lambda_{D_2}$ *if and only if* $D_1 = D_2$.

Proof: It is easy to check that Λ_D is a degree 1 representation, and so an irreducible character, of $N_{n,s}$. $N_{n,s}$ is abelian with $q^{s(n-s)}$ elements, so the number of Λ_D's is equal to the number of irreducible characters of $N_{n,s}$. The proof will be complete if we show that $\Lambda_{D_1} = \Lambda_{D_2}$ implies that $D_1 = D_2$. Now,

$$\begin{aligned}\Lambda_{D_1} = \Lambda_{D_2} &\Rightarrow \omega^{T(\mathrm{tr}\,(CD_1))} = \omega^{T(\mathrm{tr}\,(CD_2))} \text{ for all } C \in M_{n-s,s}(\mathbf{F}_q)\\ &\Rightarrow T\Big(\mathrm{tr}\,\big(C(D_1 - D_2)\big)\Big) = 0 \text{ for all } C \in M_{n-s,s}(\mathbf{F}_q)\\ &\Rightarrow \mathrm{tr}\,\big(C(D_1 - D_2)\big) = 0 \text{ for all } C \in M_{n-s,s}(\mathbf{F}_q)\end{aligned}$$

where the last implication follows from the well-known fact that $T : \mathbf{F}_q \to \mathbf{F}_p$ is non-zero.

But the last condition above implies that $D_1 - D_2 = 0$. For suppose the (i,j)–entry of $D_1 - D_2$ is non–zero. Then we can let $C = E_{ji}$, the $(n-s) \times s$ matrix with 1 in the (j,i)–position and zeros elsewhere. It is clear that for this choice of C, $\mathrm{tr}\,\big(C(D_1 - D_2)\big)$ is equal to the (i,j)–entry of $D_1 - D_2$, and so is not equal to zero, a contradiction. Thus $\Lambda_{D_1} = \Lambda_{D_2}$ implies that $D_1 = D_2$ and the proof is complete. ■

For $1 \leq s \leq n$, we can now apply the result from Lemma 1 to the normal abelian subgroup $N_{n,s}$ of U_n and its complement $H_{n,s}$. For all elements D of $M_{s,n-s}$, we let $S(D) = S_{U_n}(\Lambda_D)$.

Lemma 3. *Let* Λ_D *be an element of* $\mathrm{Irr}\,(N_{n,s})$ *for some value of* s. *Then*

$$S(D) = \{(A,\, B) \in H_{n,s} :\, ADB^{-1} = D\}.$$

Proof: Let (A, B) be any element of $H_{n,s}$. Then

$$\begin{aligned}\Lambda_D^{(A,B)}\left(\begin{pmatrix} I & 0 \\ C & I\end{pmatrix}\right) &= \Lambda_D\left(\begin{pmatrix} A & 0 \\ 0 & B\end{pmatrix}^{-1}\begin{pmatrix} I & 0 \\ C & I\end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & B\end{pmatrix}\right)\\ &= \Lambda_D\left(\begin{pmatrix} I & 0 \\ B^{-1}CA & I\end{pmatrix}\right)\\ &= \omega^{T(\mathrm{tr}\,(B^{-1}CAD))}\\ &= \omega^{T(\mathrm{tr}\,(CADB^{-1}))}\\ &= \Lambda_{ADB^{-1}}\left(\begin{pmatrix} I & 0 \\ C & I\end{pmatrix}\right)\end{aligned}$$

for all elements $\begin{pmatrix} I & 0 \\ C & I\end{pmatrix}$ of $N_{n,s}$. Thus $\Lambda_D^{(A,B)} = \Lambda_{ADB^{-1}}$ and

$$\begin{aligned} S(D) &= \left\{(A, B) \in H_{n,s} \,:\, \Lambda_D^{(A,B)} = \Lambda_D\right\}\\ &= \left\{(A, B) \in H_{n,s} \,:\, \Lambda_{ADB^{-1}} = \Lambda_D\right\}\\ &= \left\{(A, B) \in H_{n,s} \,:\, ADB^{-1} = D\right\}. \quad \blacksquare\end{aligned}$$

It follows from the proof above that $H_{n,s}$ acts on $\mathrm{Irr}(N_{n,s})$ via the action $(A, B) : \Lambda_D \mapsto \Lambda_{ADB^{-1}}$. We let $\mathcal{O}(\Lambda_D)$ be the orbit of Λ_D under this action. Of course every element $\Lambda_{D'}$ of $\mathcal{O}(\Lambda_D)$ lies under precisely the same irreducible characters of U_n as Λ_D while $S(D') \cong S(D)$.

Lemma 4. *Let $\Gamma \in \mathrm{Irr}(U_n)$ and choose t maximal such that $W_{n,t+1} \leq \ker(\Gamma)$ but $W_{n,t} \not\leq \ker(\Gamma)$. Then there exists $Y \in W_{n,t}$ such that $Y \notin \ker(\Gamma)$ and Y has precisely one non-zero entry below the main diagonal.*

Proof: Let $X = (x_{ij})$ be an element of $W_{n,t} - \ker(\Gamma)$. We define

$$Z = I_n - \sum_{l=t+2}^{n} x_{l,l-t-1}E_{l,l-t-1}.$$

We claim that $Z \notin \ker(\Gamma)$. For if f is the representation of U which affords Γ, then $f(ZX) = I$ because $ZX \in W_{n,t+1}$. Thus $f(Z) = f(X)^{-1} \neq I$. We note that

$$\begin{aligned} Z = &(I - x_{t+2,1}E_{t+2,1})(I - x_{t+3,2}E_{t+3,2}) \dots \\ &(I - x_{n,n-t-1}E_{n,n-t-1}). \end{aligned}$$

If every term in this product were an element of $\ker(f)$, then Z would also be an element of $\ker(f)$. Thus for at least one choice of k satisfying $t+2 \leq k \leq n$,

$$I - x_{k,k-t-1}E_{k,k-t-1} \notin \ker(\Gamma),$$

as required. ■

Theorem 5. *Take* $\Gamma \in \mathrm{Irr}(U_n)$ *and choose* t *maximal such that* $W_{n,t+1} \leq \ker(\Gamma)$ *but* $W_{n,t} \not\leq \ker(\Gamma)$. *Then* $\Gamma(1) \geq q^t$.

Proof: We choose $Y = I + \lambda E_{k,k-t-1}$ not in $\ker(\Gamma)$. We note that Y is an element of $N_{n,k-1}$. Thus there is at least one element Λ_D of $\mathrm{Irr}(N_{n,k-1})$ for which Y is not an element of $\ker(\Lambda_D)$, i.e. $\mathrm{tr}\, \lambda E_{1,k-t-1}D \neq 0$, where D is a $(k-1) \times (n-k+1)$ matrix over $\mathbf{F}_q$. If $D = (d_{ij})$, then it follows that $d_{k-t-1,1} \neq 0$. We let $\alpha = d_{b1}$ be the first non–zero entry in column 1 of D. Then $b \leq k-t-1$. (In fact it must be the case that $b = k-t-1$, but we do not need to prove this.) We let

$$A = I - \sum_{l=b+1}^{n} \frac{d_{l1}}{\alpha} E_{lb}.$$

We set $D' = AD$. Then $\Lambda_{D'}$ is an element of $\mathcal{O}(\Lambda_D)$ and

$$\begin{aligned} D' &= AD \\ &= \left(I - \sum_{l=b+1}^{n} \frac{d_{l1}}{\alpha} E_{lb}\right)\left(\alpha E_{b1} + \sum_{l=b+1}^{n} d_{l1}E_{l1} + \sum_{\substack{1 \leq i \leq k-1 \\ 2 \leq j \leq n-k+1}} d_{ij}E_{ij}\right) \\ &= \alpha E_{b1} + \sum_{\substack{1 \leq i \leq k-1 \\ 2 \leq j \leq n-k+1}} d'_{ij}E_{ij}, \end{aligned}$$

for some d'_{ij}'s in $\mathbf{F}_q$. Thus D' has precisely one non–zero entry in column 1, namely α in the $(b,\,1)$–position.

Now we have

$$S(D) \cong S(D') = \{(A,\,B) \in H_{n,k-1} : AD'B^{-1} = D'\}.$$

Writing $A = (a_{ij})$, the equation $AD' = D'B$ gives us relations which the $k-1-b$ entries $a_{b+1,b},\, a_{b+2,b},\, \dots,\, a_{k-1,b}$ must satisfy. Thus

$$\begin{aligned} |S(D)| = |S(D')| &\le \frac{|H_{n,k-1}|}{q^{k-1-b}} \\ &\le \frac{|H_{n,k-1}|}{q^{k-1-(k-t-1)}} \\ &= \frac{|H_{n,k-1}|}{q^t}. \end{aligned}$$

By Lemma 1, there exists $\Psi \in \mathrm{Irr}\big(S(D)\big)$ such that

$$\begin{aligned} \Gamma(1) &= \Psi(1)\frac{|H_{n,k-1}|}{|S(D)|} \\ &\ge \frac{|H_{n,k-1}|}{|H_{n,k-1}|/q^t} \\ &= q^t. \qquad \blacksquare \end{aligned}$$

Definition: For a finite group G and a positive integer l, we define $C(G,\,l)$ to be the number of irreducible characters of G having degree l.

Corollary 6. *Choose t such that $1 \le t \le n-1$. Then for $0 \le a \le t-1$, we have*

$$C(U_n,\,q^a) = C(V_{n,t},\,q^a).$$

Proof: If Γ is an element of $\mathrm{Irr}(U_n)$ such that $\Gamma(1) \le q^{t-1}$, then by Theorem 5 $W_{n,t} \le \ker(\Gamma)$. Thus Γ is an element of $\mathrm{Irr}(V_{n,t})$. Conversely, given any element Θ of $\mathrm{Irr}(V_{n,t})$, we can extend Θ to an element of $\mathrm{Irr}(U_n)$ by letting $\Theta(A) = \Theta(I)$ for all elements A of $W_{n,t}$. ■

To find the number of irreducible characters of U_n of degrees q and q^2, we need only calculate $C(V_{n,2},\, q)$ and $C(V_{n,3},\, q^2)$. While our main interest in $V_{n,2}$ is in the degree q characters, it turns out that for this group there is a simple closed formula describing the number of irreducible characters having any degree. The formulas we derive involve binomial coefficients. We adopt the conventions that $\binom{0}{0} = 1$ and $\binom{s}{t} = 0$ whenever $s < t$.

Theorem 7. $\{\Theta(1) \,:\, \Theta \in \mathrm{Irr}(V_{n,2})\} = \{q^a \,:\, 0 \le a \le \frac{n-1}{2}\}$ *and, for all values of a in this range,*

$$C(V_{n,2},\, q^a) = \binom{n-a-1}{a} q^{n-a-2}(q-1)^a + \binom{n-a-2}{a} q^{n-a-2}(q-1)^{a+1}.$$

Proof: We proceed by induction on n. $V_{2,2} = U_2$ has q irreducible characters, all of degree 1, as the theorem states. $V_{3,2} = U_3$ has q^2 characters of degree 1 while every other character degree is a power of q. Recall that the squares of the degrees of the irreducible characters of a finite group G add up to the order of G. It follows that $\{\Theta(1) \,:\, \Theta \in \mathrm{Irr}(V_{3,2})\} = \{1,\, q\}$ and that $C(V_{3,2},\, q) = q - 1$, which again is in agreement with the statement of the theorem.

For $n > 3$, we let

$$J_n = \Big\{(a_{ij}) + W_{n,2} \in V_{n,2} \,:\, a_{n,n-2} = a_{n,n-1} = 0\Big\}$$

and

$$K_n = \Big\{(b_{ij}) + W_{n,2} \in V_{n,2} \,:\, b_{ij} = 0 \text{ for } i \ne n\Big\}.$$

Now K_n is a normal abelian subgroup of $V_{n,2}$, having J_n as complement. For all α and β in $\mathbf{F}_q$, we define $\Delta_{\alpha,\beta} : K_n \to \mathbb{C}$ by

$$\Delta_{\alpha,\beta}\big((b_{ij}) + W_{n,2}\big) = \omega^{\alpha b_{n,n-2} + \beta b_{n,n-1}}.$$

It is easy to prove that $\mathrm{Irr}(K_n) = \{\Delta_{\alpha,\beta} \,:\, \alpha, \beta \in \mathbf{F}_q\}$ and $\Delta_{\alpha',\beta'} = \Delta_{\alpha,\beta}$ if and only if $\alpha' = \alpha$ and $\beta' = \beta$. Furthermore, for any elements $\tilde{A} = (a_{ij}) + W_{n,2}$ of J_n and $\tilde{B} = (b_{ij}) + W_{n,2}$ of K_n,

$$\begin{aligned}
\Delta^{\tilde{A}}_{\alpha,\beta}(\tilde{B}) &= \Delta^{\tilde{A}}_{\alpha,\beta}\big((b_{ij}) + W_{n,2}\big) \\
&= \Delta_{\alpha,\beta}\big(A^{-1}(b_{ij})A + W_{n,2}\big) \\
&= \Delta_{\alpha,\beta}\big(I + (b_{n,n-2} + b_{n,\,n-1}a_{n-1,n-2})E_{n,n-2} + \\
&\qquad\qquad b_{n,n-1}E_{n,n-1} + W_{n,2}\big) \\
&= \omega^{T(\alpha(b_{n,n-2}+b_{n,n-1}a_{n-1,n-2})+\beta b_{n,n-1})} \\
&= \omega^{T(\alpha b_{n,n-2}+(\beta+\alpha a_{n-1,n-2})b_{n,n-1})} \\
&= \Delta_{\alpha,(\beta+\alpha a_{n-1,n-2})}(\tilde{B}),
\end{aligned}$$

i.e. $\Delta^{\tilde{A}}_{\alpha,\beta} = \Delta_{\alpha,(\beta+\alpha\varepsilon)}$, where $\varepsilon = a_{n-1,n-2}$.

We define $S(\alpha,\,\beta) = S_{V_{n,2}}(\Delta_{\alpha,\beta})$. Then for all non-zero elements α of $\mathbf{F}_q$ and all elements β of $\mathbf{F}_q$,

$$S(0,\,\beta) = J_n \cong V_{n-1,2}$$

and

$$\begin{aligned}
S(\alpha,\beta) &\cong S(\alpha,\,0) \\
&= \Big\{(a_{ij}) + W_{n,2} \in J_n \;:\; a_{n-1,n-2} = 0\Big\} \\
&= \Big\{(a_{ij}) + W_{n,2} \in V_{n,2} \;:\; a_{n-1,n-2} = a_{n,n-1} = a_{n,n-2} = 0\Big\} \\
&\cong V_{n-2,2} \oplus (\mathbf{F}_q,\,+).
\end{aligned}$$

For all non-zero elements α of $\mathbf{F}_q$ and all elements β of $\mathbf{F}_q$, Lemma 1 tells us that

$$\begin{aligned}
&\Big\{\Gamma \in \mathrm{Irr}(V_{n,2}) \;:\; \big\langle \Gamma|_N,\, \Delta_{0,\beta}\big\rangle_N \neq 0\Big\} \\
&= \Big\{\big(\bar{\Delta}_{0,\beta}\Psi\big)^{V_{n,2}} \;:\; \Psi \in \mathrm{Irr}\big(S(0,\beta)\big)\Big\}
\end{aligned}$$

and

$$\begin{aligned}&\Big\{\Gamma \in \mathrm{Irr}(V_{n,2}) \,:\, \big\langle \Gamma|_N,\, \Delta_{\alpha,\beta}\big\rangle_N \neq 0\Big\}\\ =&\Big\{\Gamma \in \mathrm{Irr}(V_{n,2}) \,:\, \big\langle \Gamma|_N,\, \Delta_{\alpha,0}\big\rangle_N \neq 0\Big\}\\ =&\Big\{\big(\bar\Delta_{\alpha,0}\Psi\big)^{V_{n,2}} \,:\, \Psi \in \mathrm{Irr}\big(S(\alpha,\,0)\big)\Big\}.\end{aligned}$$

Lemma 1 also implies that, with the same notation as in the sets above,

$$\big(\bar\Delta_{0,\beta}\Psi\big)^{V_{n,2}}(1) = \Psi(1)$$

and

$$\big(\bar\Delta_{\alpha,0}\Psi\big)^{V_{n,2}}(1) = q\Psi(1).$$

Thus for $0 \leq a \leq \frac{n-2-1}{2} + 1$,

$$\begin{aligned}C(V_{n,2},\, q^a) =& qC(V_{n-1,2},\, q^a) + q(q-1)C(V_{n-2,2},\, q^{a-1})\\ =& q\tbinom{n-1-a-1}{a}q^{n-1-a-2}(q-1)^a +\\ & q\tbinom{n-1-a-2}{a}q^{n-1-a-2}(q-1)^{a+1} +\\ & q(q-1)\tbinom{n-2-(a-1)-1}{a-1}q^{n-2-(a-1)-2}(q-1)^{a-1} +\\ & q(q-1)\tbinom{n-2-(a-1)-2}{a-1}q^{n-2-(a-1)-2}(q-1)^{a}\\ =& \tbinom{n-a-2}{a}q^{n-a-2}(q-1)^a + \tbinom{n-a-3}{a}q^{n-a-2}(q-1)^{a+1} +\\ & \tbinom{n-a-2}{a-1}q^{n-a-2}(q-1)^a + \tbinom{n-a-3}{a-1}q^{n-a-2}(q-1)^{a+1}\\ =& \tbinom{n-a-1}{a}q^{n-a-2}(q-1)^a + \tbinom{n-a-2}{a}q^{n-a-2}(q-1)^{a+1}. \quad \blacksquare\end{aligned}$$

It seems to be much more difficult to find a formula for $C(V_{n,3},\, q^a)$ which holds for all relevant values of a. In this case, we will confine our attention to the characters of degree q^2 and state without proof our findings in this case.

Theorem 8. *We have $C(V_{4,\,3},\, q^2) = q(q-1)$ and for $n \geq 5$,*

$$\begin{aligned}C(V_{n,\,3},\, q^2) =& \tbinom{n-3}{2}q^{n-4}(q-1)^2 + \tbinom{n-4}{2}q^{n-4}(q-1)^3 +\\ & (n-3)q^{n-2}(q-1) + (n-5)q^{n-2}(q-1)^2.\end{aligned}$$

We can now provide formulae for the number of irreducible characters of U_n having the degrees 1, q and q^2.

Theorem 9. *For $n \geq 2$, U_n has q^{n-1} irreducible characters of degree* 1*. For $n \geq 3$, U_n has $(n-2)q^{n-3}(q-1)+(n-3)q^{n-3}(q-1)^2$ irreducible characters of degree q. U_4 has $q(q-1)$ irreducible characters of degree q^2; for $n \geq 5$, U_n has $\binom{n-3}{2}q^{n-4}(q-1)^2 + \binom{n-4}{2}q^{n-4}(q-1)^3 + (n-3)q^{n-2}(q-1) + (n-5)q^{n-2}(q-1)^2$ irreducible characters of degree q^2.*

Proof: This follows from Corollary 6, Theorem 7, and Theorem 8. ■

References

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