

AN UNUSUAL CONSTRUCTION OF THE FOURIER TRANSFORM

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Introduction

The usual construction of the Fourier transform involves working on $L^1(\mathbb{R})$ (eg. [1]) or the Schwartz Class \mathcal{S} of rapidly decreasing C^∞ functions (eg. [2]). The Fourier transform is then extended onto $L^2(\mathbb{R})$ by taking limits as both $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and \mathcal{S} are dense in $L^2(\mathbb{R})$.

I'd like to present an unusual construction of the Fourier transform in which we use its translation and dilation properties:

- if $g(x) = f(x + \alpha)$ then $\hat{g}(\omega) = e^{i\alpha\omega} \hat{f}(\omega)$,
- if $g(x) = f(\lambda x)$ then $\hat{g}(\omega) = 1/|\lambda| \hat{f}(\omega/\lambda)$.

This construction is in the spirit of the “multiresolution analysis” structure [3] which is used to build discrete wavelet bases [4]. However, if you don't know anything about this structure the construction is still surprisingly straightforward.

The Definition

I'll be using both \hat{f} and $\mathcal{F}f$ to denote the Fourier transform of f . For translation and dilation I'll write:

- $(\mathcal{T}_\alpha f)(x) := f(x + \alpha)$ for any $\alpha \in \mathbb{R}$,
- $(\mathcal{D}\lambda f)(x) := f(\lambda x)$ for any $\lambda \in \mathbb{R} \setminus \{0\}$,
- $(\mathcal{R}\alpha f)(x) := e^{i\alpha x} f(x)$ for any $\alpha \in \mathbb{R}$.

This allows me to write the translation and dilation properties as

$$\mathcal{F}\mathcal{T}_\alpha f = \mathcal{R}_\alpha \mathcal{F}f$$

and

$$\mathcal{F}D_\lambda f = \frac{1}{|\lambda|} D_{\frac{1}{\lambda}} \mathcal{F}f.$$

Now to define \mathcal{F} , we do the following:

1. for the characteristic function of $[0, 1)$ we define

$$\mathcal{F}(\chi_{[0,1)}) = \frac{1 - e^{-i\omega}}{i\omega},$$

2. we extend \mathcal{F} to any $\chi_{[n,n+1)}$ by using the translation rule

$$\mathcal{F}\mathcal{T}_n f = \mathcal{R}_n \mathcal{F}f$$

for all $n \in \mathbb{Z}$,

3. we further extend \mathcal{F} to functions of the form $\chi_{[\lambda^{-1}n, \lambda^{-1}(n+1))}$ by using the dilation rule

$$\mathcal{F}D_\lambda f = \frac{1}{|\lambda|} D_{\frac{1}{\lambda}} \mathcal{F}f$$

for all $\lambda \in 2\mathbb{Z}$,

4. finally we extend \mathcal{F} to the linear span of these by assuming that \mathcal{F} is linear.

If this process works, we have defined \mathcal{F} on D , the set of dyadic step functions. These are just the simple functions whose jumps occur at $n/2^m$ where $n, m \in \mathbb{Z}$. It is easy to construct a well defined function which has these properties. In fact the definition spells out a formula:

$$\begin{aligned} f(x) &= \sum_{r=-R}^R a_r \chi_{[0,1)}(2^j x - r) \Rightarrow \\ \mathcal{F}f(\omega) &= \frac{1 - e^{-i\omega/2^j}}{i\omega} \sum_{r=-R}^R a_r e^{-ir\omega/2^j}. \end{aligned}$$

Our aim was to produce \mathcal{F} on $L^2(\mathbb{R})$. Given that the set of simple functions is dense in $L^2(\mathbb{R})$ it is clear that D is also dense in

$L^2(\mathbb{R})$. So, if we can show that this function \mathcal{F} we have defined is continuous in the $L^2(\mathbb{R})$ norm then we can extend \mathcal{F} to all of $L^2(\mathbb{R})$.

This turns out to be surprisingly straight forward. Taking

$$f(x) = \sum_{r=-R}^R a_r \chi_{[0,1)}(2^j x - r)$$

we see that $\|f\|_2^2 = \sum_{r=-R}^R |a_r|^2 / 2^j$.

Now we have to find $\|\mathcal{F}f\|_2^2$. Using our formula above and the definition of the norm:

$$\begin{aligned} \|\mathcal{F}(f)\|_2^2 &= \\ & \int_{-\infty}^{\infty} \left| \frac{1 - e^{-i\frac{\omega}{2^j}}}{i\omega} \right|^2 \left(\sum_{k=-N}^N a_k e^{-ik\frac{\omega}{2^j}} \right) \left(\sum_{l=-N}^N \overline{a_l} e^{il\frac{\omega}{2^j}} \right) d\omega = \\ & \int_{-\infty}^{\infty} \frac{2(1 - \cos \frac{\omega}{2^j})}{\omega^2} \left[\left(\sum_{k=-N}^N |a_k|^2 \right) + \left(\sum_{k \neq l} a_k \overline{a_l} e^{-i(k-l)\frac{\omega}{2^j}} \right) \right] d\omega. \end{aligned}$$

So we need to evaluate:

$$\int_{-\infty}^{\infty} \frac{2(1 - \cos \omega)}{\omega^2} e^{ir\omega} d\omega,$$

for $r \in \mathbb{Z}$. This is an easy piece of contour integration, giving 2π if $r = 0$ and zero otherwise. Filling this in we see:

$$\|\mathcal{F}f\|_2^2 = \sum_{r=-R}^R |a_r|^2 / 2^j 2\pi = 2\pi \|f\|_2^2.$$

So, not only is \mathcal{F} continuous but it just scales the norm. This means that we may extend \mathcal{F} to a continuous map from $L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ which preserves the inner product:

$$(f, g) = 2\pi(\mathcal{F}f, \mathcal{F}g).$$

What now?

Note that we could show that \mathcal{F} as defined on D also extends to a continuous map $\mathcal{F} : L^1(\mathbb{R}) \rightarrow L^\infty(\mathbb{R})$ by examining the $L^1(\mathbb{R})$ norm of f and the $L^\infty(\mathbb{R})$ norm of $\mathcal{F}f$. This might motivate us to try to get the usual integral formula for the Fourier transform back again.

This can be done for $f \in D$ by first considering f as a function with steps of width 2^{-j} , and then splitting each step in half to get the same function written in terms of steps of width $2^{-(j+1)}$. This turns our formula for \mathcal{F} into a Riemann sum for the integral:

$$(\mathcal{F}f)(\omega) = \int f(x)e^{-i\omega x} dx \quad f \in D.$$

This can naturally be extended to suitable sets larger than D .

By looking at the dilation and translation relations carefully (or by using the integral formula) we get extended translation and dilation, this time for all $f \in L^2(\mathbb{R})$, $\alpha \in \mathbb{R}$ and $\lambda \in \mathbb{R} \setminus \{0\}$:

- $\mathcal{F}\mathcal{T}_\alpha f = \mathcal{R}_\alpha \mathcal{F}f$,
- $\mathcal{F}\mathcal{D}_\lambda f = \frac{1}{|\lambda|} \mathcal{D}_{\frac{1}{\lambda}} \mathcal{F}f$,
- $\mathcal{F}\mathcal{R}_\alpha f = \mathcal{T}_{-\alpha} \mathcal{F}f$.

This provides us with a neat way to show that \mathcal{F} is invertible on $L^2(\mathbb{R})$. Suppose we defined \mathcal{G} with the translation and dilation properties we expect of \mathcal{F}^{-1} . Then by proceeding as we did for \mathcal{F} , we arrive at an integral formula and the following properties for \mathcal{G} :

- $\mathcal{G}\mathcal{T}_\alpha f = \mathcal{R}_{-\alpha} \mathcal{G}f$,
- $\mathcal{G}\mathcal{D}_\lambda f = \frac{1}{|\lambda|} \mathcal{D}_{\frac{1}{\lambda}} \mathcal{G}f$,
- $\mathcal{G}\mathcal{R}_\alpha f = \mathcal{T}_\alpha \mathcal{G}f$.

We examine $\mathcal{I} = \mathcal{F}\mathcal{G}$ and how it interacts with \mathcal{T}_n , \mathcal{D}_λ and $\chi_{[0,1]}$. Using the algebraic properties of \mathcal{F} and \mathcal{G} :

$$\mathcal{I}\mathcal{T}_n f = \mathcal{F}\mathcal{G}\mathcal{T}_n f = \mathcal{F}\mathcal{R}_{-n} \mathcal{G}f = \mathcal{T}_n \mathcal{F}\mathcal{G}f = \mathcal{T}_n \mathcal{I}f$$

$$\mathcal{I}\mathcal{D}_\lambda f = \mathcal{F}\mathcal{G}\mathcal{D}_\lambda f = \frac{1}{|\lambda|} \mathcal{F}\mathcal{D}_{\frac{1}{\lambda}} \mathcal{G}f = \frac{|\lambda|}{|\lambda|} \mathcal{D}_\lambda \mathcal{F}\mathcal{G}f = \mathcal{D}_\lambda \mathcal{I}f.$$

Thus \mathcal{I} commutes with \mathcal{T}_n and \mathcal{D}_λ , so if we can determine the image of $\chi_{[0,1]}$ we can determine the image of D . Using the integral formula and a little contour integration we see:

$$\begin{aligned} (\mathcal{I}\chi_{[0,1]})(x) &= (\mathcal{F}\mathcal{G}\chi_{[0,1]})(x) \\ &= \mathcal{F}\left(\frac{e^{i\omega} - 1}{i\omega}\right) \\ &= \int \frac{e^{i\omega} - 1}{i\omega} e^{-i\omega x} d\omega = 2\pi\chi_{[0,1]}(x) \end{aligned}$$

for almost every x . So \mathcal{I} acts on D by multiplying by 2π . Using the fact that \mathcal{I} is continuous we see that \mathcal{I} acts on all of $L^2(\mathbb{R})$ in this way, and so $(2\pi)^{-1}\mathcal{G}$ is a right inverse for \mathcal{F} . Naturally a similar argument shows that it is also a left inverse.

To finish up

This is a curious construction of the Fourier transform. It is even quite easy to extend it to $L^2(\mathbb{R}^n)$. One interesting point I didn't touch on is that we may change the first rule with which we defined \mathcal{F} from:

- for the characteristic function of $[0, 1)$ we define

$$\mathcal{F}(\chi_{[0,1)}) = \frac{1 - e^{-i\omega}}{i\omega},$$

to the seemingly weaker:

- $\mathcal{F}(\chi_{[0,1)})$ is continuous at zero and has value 1 at zero.

This is because $\chi_{[0,1)}$ satisfies the dilation equation:

$$\chi_{[0,1)}(x) = \chi_{[0,1)}(2x) + \chi_{[0,1)}(2x - 1),$$

but that is another story, [5].

References

[1] Elias M. Stein and Guido Weiss, Introduction to Fourier Analysis on Euclidean Spaces. Princeton University Press: 1971.

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- [2] H. Dym and H. P. McKean, *Fourier Series and Integrals*. Academic Press: 1972.
 - [3] Yves Meyer, *Wavelets and Operators*. Cambridge University Press: 1992.
 - [4] Ingrid Daubechies, *Ten Lectures on Wavelets*. SIAM: (1992).
 - [5] David Malone, *Fourier analysis, multiresolution analysis and dilation equations*, Master's thesis (1997), University of Dublin.

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