A SPECTRAL PROPERTY FOR THE SYMBOLIC DYNAMICAL SYSTEM

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Let (X, ϕ) be a topological dynamical system, that is, X is a compact Hausdorff space and $\phi : X \to X$ is a homeomorphism. Recently, a great deal of attention has been paid to topological dynamical systems with the so called chaotic behaviour, see [4, 6] and the bibliography therein for a comprehensive discussion about what the chaotic behaviour means mathematically.

We can consider the linear operator T_ϕ defined by $T_\phi(f) = f \circ \phi$, where f is a function on X. If (X, ϕ) and (Y, ψ) are topological dynamical systems, then they are *isomorphic*, and we write $\phi \cong \psi$, if there exists a homeomorphism $h : X \to Y$ such that $\phi = h^{-1}\psi h$. Clearly, $T_\phi = T_h T_\psi T_{h^{-1}}$ and therefore the spectral properties of T_ϕ are natural invariants with respect to the isomorphism relationship. In particular, if the spectral properties of T_ϕ are distinct from those of T_ψ, then we have $\phi \ncong \psi$. It is interesting to investigate spectral properties of the operators T_ϕ that are typical for the dynamical systems (X, ϕ) with chaotic behaviour and this is the principal topic of the paper.

Consider the two-element group $\mathbf{Z}_2 = \{0, 1\}$ under addition modulo 2 and define the set $\Omega = \mathbf{Z}_2^{\mathbf{Z}}$; that is, Ω is the set of all doubly infinite sequences $\omega = (\omega_n)_{n \in \mathbf{Z}}$ with $\omega_n \in \mathbf{Z}_2$. Obviously Ω is compact when endowed with the product topology and operation. Define the *shift transformation* $\phi : \Omega \to \Omega$ by setting $\phi(\omega) = (\omega')$, where $\omega'_n = \omega_{n+1}$ for all $n \in \mathbf{Z}$. The dynamical system (Ω, ϕ) is called the *symbolic dynamical system.* This is a traditional first choice for the system exemplifying chaotic behaviour [3]. Moreover, sometimes a dynamical system (X, ψ) is said to have chaotic behavior if there exists an invariant subset

$Y \subseteq X$ such that the dynamical system $(Y, \psi|_Y)$ is isomorphic to the dynamical system (Ω, ϕ) [4].

Theorem 1.
(a) *There exists a continuous function* $g_0 \in C(\Omega)$ *such that for each complex* λ, $|\lambda| = 1$ *the open ball* $B(g_0, 1/8)$ *does not intersect the range of* $T_\phi - \lambda I$.
(b) 1 *is the only eigenvalue of* T_ϕ.

The property (a) of the operator T_ϕ from Theorem 1 contrasts with the properties of the operator T_ϕ in $L_2(\mu)$ with the Bernoulli invariant measure μ, [5]: for each $\lambda \in \mathbf{T}$ the range of the operator $\lambda I - T_\phi$ is dense in $L_2(\mu)$. Note also that the property mentioned in the theorem is invariant with respect to isomorphism:

Corollary 2. *Let* (Y, ψ) *be a topological dynamical system that is isomorphic to* (Ω, ϕ). *Then the following spectral properties are valid:*
(a) *there exists a continuous function* $\tilde{g}_0 \in C(Y)$ *such that for each* $\lambda \in \mathbf{T}$, *the open ball* $B(\tilde{g}_0, 1/8)$ *does not intersect the range of* $T_\psi - \lambda I$;
(b) 1 *is the only eigenvalue of* T_ψ.

The *combination* of spectral properties mentioned in Theorem 1 looks rather peculiar and may be typical only for dynamical systems which behave in a similar way to the hyperbolic homeomorphisms, [3]. This suggests that the following definition could be useful.

Definition. A topological dynamical systems (X, ψ) is said to be *s-chaotic* if the operator $T_\psi : C(X) \to C(X)$ has no eigenvalues apart from 1 and, on the other hand, there exists an open ball $B(g_0, \varepsilon) \subset C(X)$ satisfying

$$B(g_0, \varepsilon) \bigcap \Big(\bigcup_{\lambda \in \mathbf{T}} (T_\psi - \lambda I)(C(X)) \Big) = \emptyset.$$

Note that by Corollary 2 this definition is in line with the definition of chaotic behaviour as suggested in [4]. The tentative analysis shows that this definition does not apply to the dynamical systems

(Y, ψ) which are traditionally considered as 'non-chaotic' or have a non-chaotic component.

Example 3. As an example consider the irrational rotation ψ_α of a circle:

$$e^{i\theta} \mapsto e^{i(\theta+2\pi\alpha)}$$

where $0 \leq \theta < 2\pi$ and α is a fixed irrational real number. This mapping is a typical example of an ergodic but not chaotic mapping. The spectrum of the operator T_{ψ_α} coincides with the unit circle of the complex plane, but its further properties contrast sharply with those mentioned in the definition of s-chaotic behaviour: the operator T_{ψ_α} has a countable number of eigenvalues $\lambda_k = e^{i2\pi\alpha k}$, $k = 0, \pm 1, \pm 2, \dots$ and for any $\lambda \neq \lambda_k$, $k = 0, \pm 1, \pm 2, \dots$ the range of the operator $T_{\psi_\alpha} - \lambda I$ is dense in $C(\mathbf{T})$.

On the other hand, the chaotic homeomorphisms different from the symbolic dynamical system are likely to be s-chaotic.

Problem. Prove that an algebraic toral automorphism [4] is s-chaotic.

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