# GROUPS WITH A SUBLINEAR ISOPERIMETRIC INEQUALITY

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**Abstract** We give a proof that if a finitely presented group $G$ admits a presentation with a sublinear isoperimetric inequality, then $G$ is either free or finite.

## 1. Introduction

Let $G = \langle S|R \rangle$ be a finitely generated group. Then a word in $F(S)$, the free group on $S$, is equal to the identity in $G$ if and only if there exist words $u_i$ in $S$ for $1 \leq i \leq n$ such that

$$w = \prod_{i=1}^{n} u_i r_i u_i^{-1} \text{ as reduced words,}$$

where for all $i$ with $1 \leq i \leq n$ either $r_i \in R$ or $r_i^{-1} \in R$.

**Definition 1.1** With $G$ as above, let $w$ be a word in $S$ which is equal to the identity in $G$. Then the **area** of $w$, $A(w)$ is defined to be

$$\min\{n \in \mathbb{N} \mid \exists \text{ an equality } w = \prod_{i=1}^{n} u_i r_i u_i^{-1} \text{ in } F(S)\}.$$

We do not work with this definition of area but rather with a more geometric formulation.

**Definition 1.2** A **map** is a finite, planar, oriented, connected and simply connected combinatorial 2-complex.

Let $M$ be a map with edge set $E(M)$. If $e = (v_1, v_2) \in E(M)$ then we write $\tilde{e}$ for the edge $(v_2, v_1)$.

**Definition 1.3** A **paired alphabet** is a finite set $S$ together with an involution $f : S \to S$. We usually write $f(s) = s^{-1}$.

For example, an inverse closed set of generators of a group is a paired alphabet, where the involution is the group inverse.

**Definition** A **diagram** over a paired alphabet $S$ is a triple $(M, S, l)$ where $M$ is a map, $S$ is a paired alphabet and $l : E(M) \to S$ satisfies $l(\tilde{e}) = (l(e))^{-1}$ for all $e \in E(M)$. If $e \in E(M)$ then $l(e)$ is called the **label** of $e$.

When we refer to a **path** in a graph $X$ we mean a finite sequence of adjacent edges. If the sequence of terminal vertices of edges consists of distinct vertices then we call the path **simple**. A **loop** is a path such that the terminal vertex of the final edge equals the initial vertex of the first. Thus a **simple loop** is a loop which is simple as a path. If $X$ is the 1-skeleton of a diagram $(M, S, l)$ and $p = e_1, \ldots, e_n$ is a path in $X$ we define its label $l(e)$ to be the word $l(e_1) \cdots l(e_n)$ in $S$. If $f$ is a face of $M$ we denote its boundary loop by $\partial f$ and write $l(f)$ for $l(\partial f)$.

**Definition 1.5** Let $G = \langle S, R \rangle$ be a finitely presented group where $S$ is an inverse closed generating set for $G$. A **van Kampen diagram** over $G$ is a diagram $M$ over $S$ such that for all faces $f$ of $M$, $l(f) = r^{\pm 1}$ for some relator $r \in R$. The **area** of such a diagram is the number of its faces.

The hypotheses on a map $M$ ensure that its boundary $\partial M$ is a loop. We write $l(M)$ for $l(\partial M)$. If $G = \langle S | R \rangle$ is a group presentation and $w$ is a word in $S$ then we write $\bar{w}$ for the element of $G$ represented by $w$.

**Lemma 1.6 (van Kampen)** Let $G = \langle S | R \rangle$ be a finitely presented group and let $w$ be a word in $S$. Then $\overline{w} = 1_G$ if and only if there exists a Van Kampen diagram $M$ over $G$ with $l(M) = w$. Moreover $A(w)$ is equal to the least area of a van Kampen diagram for $w$.

*Proof*: See [7]. ■

**Definition 1.7** Let $G = \langle S | R \rangle$ be a finitely presented group. Then the **Dehn function** $D$ of $G$ with respect to $S$ and $R$ is the function $D : \mathbb{N} \to \mathbb{N}$ given by

$$D(n) = \max\{A(w) \mid \overline{w} = 1_G \text{ and } l(w) \leq n\}.$$

We say that $G = \langle S | R \rangle$ satisfies a **linear isoperimetric inequality** if its Dehn function is $O(n)$, i.e. there exists $C \geq 0$ such that for all $n \in \mathbb{N}$, $D(n) \leq Cn$. If

$$\lim_{n \to \infty} \left( \frac{D(n)}{n} \right) = 0$$

then we say that $G$ satisfies a **sublinear isoperimetric inequality** and if

$$\lim_{n \to \infty} \left( \frac{D(n)}{n^2} \right) = 0$$

then we say that $G$ satisfies a **subquadratic isoperimetric inequality**. If $G$ is a finitely presented group then it is well known that the following are equivalent.

1. $G$ is hyperbolic in the sense of Gromov [3].
2. $G$ satisfies a linear isoperimetric inequality.
3. $G$ satisfies a subquadratic isoperimetric inequality.

The proof that the first statement is equivalent to the second can be found in [6]. The second clearly implies the third. The fact that a subquadratic isoperimetric inequality implies a linear one is originally due to Gromov [3] and can be found in [2], [4] and [5]. In particular we see that the satisfaction of a sublinear isoperimetric inequality is invariant under quasi–isometry.

### Quasi-Trees

A graph $X$ is said to be of **bounded valency** if there exists an integer $N$ such that the valency of every vertex of $X$ is at most $N$.

**Definition 2.1** Let $Q$ be a connected graph of bounded valency. We call $Q$ a $K$–**quasi-tree** if every simple loop in $Q$ has length at most $K$. If there exists a non-negative integer $K$ for which $Q$ is a $K$–quasi-tree then we call $Q$ a **quasi-tree**.

**Theorem 2.2** *A finitely generated group $G$ acts freely on a quasi–tree if and only if $G$ is isomorphic to a free product of free groups and finite groups.*

A proof is given in [1].

### The Main Result

Let $G$ be a group with a finite generating set $S$. We write $\Gamma_S(G)$ for the Cayley graph of $G$ with respect to $S$. Suppose that we have a sublinear isoperimetric inequality amongst the simple loops in $\Gamma_S(G)$ (i.e. in formation of the Dehn function we only consider simple loops). In this situation we say that $G$ satisfies a sublinear **simple isoperimetric inequality**.

**Proposition 3.1** *If a finitely presented group $G$ satisfies a sublinear simple isoperimetric inequality, then there is a bound on the length of simple loops in its Cayley graph.*

*Proof:* Suppose that in the Cayley graph $\Gamma_S(G)$ of $G$ with respect to some finite presentation $\langle S|R\rangle$ there is satisfied a sublinear simple isoperimetric inequality. Let $K$ be the maximum length of the relators. Assume that $R$ is not empty (if it is then the theorem follows easily), so that $K \geq 1$. If a simple loop $\Lambda$ in $\Gamma_S(G)$ has length $l$ then the number of relators we require to fill $\Lambda$ is at least the next integer after $\frac{l}{K}$. So unless there is a bound on the length of simple loops in $\Gamma_S(G)$, the best bound below for the isoperimetric inequality of $G$ is at least a linear function. ■

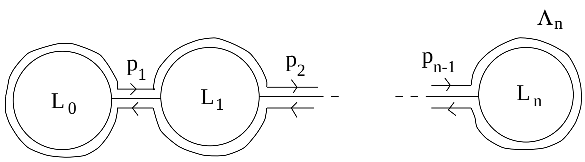
**Corollary 3.2** *A finitely presented group $G$ admits a sublinear simple isoperimetric inequality for some finite presentation if and only if $G$ is quasi-free.*

*Proof:* If $G$ is a quasi–free group, then with respect to the standard generating set there is a bound on the length of simple loops in its Cayley graph. So $G$ clearly satisfies a sublinear simple isoperimetric inequality.

Conversely, by Proposition 3.1, the sublinear simple isoperimetric inequality gives us a bound on the length of simple loops in $\Gamma_S(G)$. Hence $\Gamma_S(G)$ is a quasi-tree upon which $G$ acts freely. Now the result follows by Theorem 2.2. ■

Note that the property of whether or not a group admits a sublinear isoperimetric inequality is not invariant under quasi-isometry. For example, $\mathbb{Z}$ trivially satisfies a sublinear isoperimetric inequality with respect to the standard generating set. However, with the standard generating set the group $\mathbb{Z} \oplus \mathbb{Z}_3$, which is quasi–isometric to $\mathbb{Z}$, does not.

In what follows we use the notation $i(p)$ and $t(p)$ to denote the initial and terminal vertices of a path $p$ in a graph $X$. We also give each loop $L$ in $X$ a preferred orientation and if $v_1$ and $v_2$ are vertices of $L$ write $L(v_1, v_2)$ for the path obtained when travelling around $L$ from $v_1$ to $v_2$ in a positive direction.




*A loop with linear area.*

**Theorem 3.3.** *If a group $G$ has a finite presentation $\langle S|R\rangle$ with respect to which $G$ satisfies a sublinear isoperimetric inequality then $G$ is either free or finite.*

*Proof:* In particular, $G$ satisfies a sublinear simple isoperimetric inequality. Hence by Corollary 3.2, $\Gamma_S(G)$ is a quasi–tree and $G$ is quasi–free. Now suppose that there is a nontrivial finite group $H$ which is a free factor of $G$. Let $H_0$ be the subgraph of $\Gamma_S(G)$ induced by the vertex set of $H$. Either $G$ is finite or $\Gamma_S(G)$ contains infinitely many copies of $H_0$. Let $M$ be the maximum length of a relator in $R$. We may choose a loop $L$ of strictly positive area in $H$ and copies $H_0 = H$, $H_1, \ldots, H_n, \ldots$ of $H$ containing copies $L_0 = L$, $L_1, \ldots, L_n, \ldots$ of $L$ in such a way that there are paths $p_1, \ldots, p_n, \ldots$, all of the same length, where for each $j$, $p_j$ goes from $L_{j-1}$ to $L_j$ as in figure 1. Furthermore, we may choose these such that $d(i(p_j), t(p_j)) > M$ for all $j$. Let $\Lambda_n$ be the loop

$$L_0(1, i(p_1)) * p_1 * L_1(t(p_1), i(p_2)) * p_2 * \cdots$$
$$*p_{n-1} * L_n * p_{n-1}^{-1} * \cdots * p_2^{-1} * L_1(i(p_2), t(p_1)) * p_1^{-1} * L_0(i(p), 1).$$

Let $N = A(L)$. Then since $i(p_j)$ and $t(p_j)$ are cut points of $\Gamma_S(G)$ and the endpoints of the paths $p_j$ are sufficiently far apart, $A(\Lambda_n) = Nn$. Thus $G$ satisfies an isoperimetric inequality which is at least linear, a contradiction. Hence $G$ is free. ■

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