Outline Solutions of the Problems for the 39th IMO

1. Choose a coordinate system so that AC and BD are the axes and the points A, B, C, D have coordinates $(0, a)$, $(b, 0)$, $(0, c)$ and $(d, 0)$, respectively. The equations of the perpendicular bisectors of AB and DC are

$$2bx - 2ay = b^2 - a^2 \text{ and } 2dx - 2cy = d^2 - c^2,$$

respectively. The coordinates of P are obtained by solving these equations. So, P has coordinates (p, q), where

$$p = (a(d^2 - c^2) - c(b^2 - a^2))/2(ad - bc)$$

and

$$q = (b(d^2 - c^2) - d(b^2 - a^2))/2(ad - bc).$$

Since the triangles ABP and PCD have the same orientation, they have equal areas if and only if

$$-ab + bq + pa = pc - cd + dq,$$

using the formula for the area of a triangle in terms of the coordinates of its vertices. Using the expressions for p and q above, this condition reduces to

$$(ac - bd)((a - c)^2 + (b - d)^2) = 0.$$

Since a and c have opposite signs, we see that the triangles ABP and PCD have the same area if and only if $ac = bd$. The last condition holds if and only if the triangles ABE and CDE are similar, where E is the point of intersection of AC and BD. It is clear that this last condition holds if and only if $ABCD$ is a cyclic quadrilateral.

2. Consider one contestant. Suppose he receives a "pass" from r judges. Then he receives a "fail" from $b - r$ judges. So, the number of pairs of judges who agree for this contestant is

$$\begin{aligned}\binom{r}{2}+\binom{b-r}{2} &= \frac{1}{2}(r^2+(b-r)^2-b)\\ &\geq \frac{1}{2}(\frac{1}{2}(r+b-r)^2-b)=\frac{1}{4}((b-1)^2-1).\end{aligned}$$

Thus

$$\binom{r}{2}+\binom{b-r}{2}\geq\frac{1}{4}(b-1)^2.$$

Since there are $\binom{b}{2}$ pairs of judges and each pair agrees on at most k contestants, the total number of agreements is at most $k\binom{b}{2}$. Since there are a contestants, we get

$$k\binom{b}{2}\geq\frac{a}{4}(b-1)^2.$$

So, $kb\geq a(b-1)/2$ and the result follows.

3. (Solution by Arkady Slinko.) It is easy to see that d is a multiplicative function and that $d(p^r)=r+1$ when p is a prime. So, if the prime factorization of n is $p_1^{r_1}\cdots p_m^{r_m}$, then

$$\frac{d(n^2)}{d(n)}=\frac{(2r_1+1)\cdots(2r_m+1)}{(r_1+1)\cdots(r_m+1)}. \qquad (*)$$

Let k be a positive integer of the form $(*)$. Clearly, k is odd and, when $n=1$, we get $k=1$. We claim that every odd integer can be represented in the form $(*)$. We prove this by induction. Suppose that all odd $k_0<k$ are representable in the form $(*)$. Let $k=2^sk_0-1$, where s and k_0 are positive integers and k_0 is odd. Since k_0 is representable, it suffices to prove that $(2^sk_0-1)/k_0$ can be represented in the form $(*)$. Letting $m=(2^s-1)k_0$, we get

$$\begin{aligned}\frac{2^sk_0-1}{k_0} &= \frac{2^sm-(2^s-1)}{m}\\ &= \frac{2m-1}{m}\times\frac{4m-3}{2m-1}\times\cdots\times\frac{2^sm-(2^s-1)}{2^{s-1}m-(2^{s-1}-1)},\end{aligned}$$

a telescoping product of the form $(*)$. The result follows.

4. Since

$$b(a^2b+a+b)-a(ab^2+b+7)=b^2-7a,$$

if ab^2+a+b is divisible by ab^2+b+7, then so is b^2-7a. Now $b^2-7a<ab^2+b+7$ and so, if $b^2-7a\geq 0$, we get b^2-7a=0. Thus $(a,b)=(7t^2,7t)$, for some positive integer t. On the other hand, if $(a,b)=(7t^2,7t)$, for some positive integer t, then $ab^2+b+7=7(49t^4+t+1)$ divides $a^2b+a+b=7t(49t^4+t+1)$.

Suppose now that $b^2-7a<0$. Then ab^2+b+7 divides the positive integer $7a-b^2$. If $b\geq 3$, then $ab^2+b+7>9a$. Hence $b=1$ or $b=2$. If $b=1$, then $a+8$ divides $7a-1$ and, since the multiple is at most 6, we get $a=11$ or $a=49$. It is easy to see that $(a,b)=(11,1),(49,1)$ are solutions to the problem. Finally, if $b=2$, then $4a+9$ divides $7a-4$ and since the only possibility is $4a+9=7a-4$, we do not get an integer value for a in this case.

So, the complete list of solutions is:

$$(11,1),(49,1),(7t^2,7t),\ \text{for all } t\in\mathbb{N}.$$

5. (Solution by Andy Liu.) It is easy to see that

$$\angle RMB=\angle AML=\angle MKL=\angle BSK$$

and that

$$\angle RBM=\angle BMK=\angle BKM=\angle BSK.$$

So the triangles MRB and KBS are similar. Thus $BR\times BS=BK^2$. Since $\angle RBM=\angle SBK$ and IB bisects $\angle ABC$, the lines IB and RS are perpendicular. Thus

$$\begin{aligned}IR^2+IS^2&=RB^2+BS^2+2IB^2\\&>RB^2+BS^2+2BK^2\\&=RB^2+BS^2+2RB\times BS=RS^2\end{aligned}$$

So, since $IR^2+IS^2>RS^2$, the angle RIS is acute.

6. Let $f(n^2(f(m)) = mf(n)^2$ for all m and n in $\mathbb{N}$. Let $f(1) = a$. Then

$$f(f(m)) = a^2m \text{ and } f(an^2) = f(n)^2$$

for all $m, n \in \mathbb{N}$. If $f(r) = f(s)$, then $a^2r = f(f(r)) = f(f(s)) = a^2s$ and thus $r = s$, so that f is injective. Note that $f(a) = f(f(1)) = a^2$.

Now

$$\begin{aligned}(f(m)f(n))^2 = f(m)^2f(n)^2 &= f(am^2)f(n)^2 = f(n^2f(f(am^2))) \\ &= f(a^3m^2n^2) = f(a(amn)^2) = f(amn)^2.\end{aligned}$$

So, $f(amn) = f(m)f(n)$. In particular, $f(an) = af(n)$. Thus

$$af(mn) = f(m)f(n) \text{ for all } m, n \in \mathbb{N}.$$

We claim that a divides $f(m)$ for all positive integers m. For suppose that $a \neq 1$ and let p^k be the highest power of the prime p that divides a. Since $af(m^2) = f(m)^2$, p^r divides $f(m)$ for all m, for some integer $r \geq k/2$. Consider the largest such r. Then p^{k+r} divides $f(m)^2$, for all $m \in \mathbb{N}$. So $k + r \leq 2r$. Thus $r \geq k$ and hence p^k divides $f(m)$ for all m. Thus a divides $f(m)$ for all m.

So, there exists a function $g : \mathbb{N} \to \mathbb{N}$ such that $f(n) = ag(n)$ for all n. Now $g(1) = 1$ and $g(n^2g(m)) = mg(n)^2$ for all $m, n \in \mathbb{N}$. Since $g(1998) \leq f(1998)$, the least possible value of $f(1998)$ for functions satisfying the given identity will be attained for a function f with $f(1) = 1$.

Thus, we need only consider functions $f : \mathbb{N} \to \mathbb{N}$ such that

$$f(1) = 1 \text{ and } f(n^2f(m)) = mf(n)^2$$

for all $m, n \in \mathbb{N}$. Then $f(mn) = f(m)f(n)$ and $f(f(m)) = m$ for all positive integers m and n. Clearly, a function f with the last two properties also satisfies $f(1) = 1$ and $f(n^2f(m)) = mf(n)^2$ for all positive integers m and n. By the multiplicative property, once $f(p)$ is determined for each prime p, then f will be completely determined. Let p be a prime and suppose that $f(p) = ab$ for certain $a, b \in \mathbb{N}$. Then $p = f(f(p)) = f(ab) = f(a)f(b)$. We may suppose that $f(a) = 1$. Then since $f(1) = 1$, the injectivity of f implies that $a = 1$. Thus $f(p)$ is a prime for all primes p and injectivity implies that f maps distinct primes to distinct primes. Since $1998 = 2 \times 3^3 \times 37$, to find a function satisfying the given conditions such that $f(1998)$ is as small as possible, it is clear that letting

$$f(2) = 3, \quad f(3) = 2, \quad f(37) = 5$$

and for each other prime p, $f(p)$ equal any prime not already chosen, will produce the required function. Thus the minimal possible value for $f(1998)$ is $3 \times 2^3 \times 5 = 120$.