THE BAER-SPECKER GROUP

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The **Baer-Specker group**, **P**, is the group of functions from the natural numbers **N** into the integers **Z**. While **P** is very easy to define, it is the source of a wealth of problems, some of which have only recently been solved. The aim of this paper is to present an introductory account of old and new results about **P**, and to explore some of the connections of this group with other areas of mathematics, in particular with the real numbers, infinitary logic, and combinatorial set theory.

Introduction

The Baer-Specker group **P** is an infinite abelian group under the addition

$$(f + g)(n) = f(n) + g(n)$$

for $n \in \mathbf{N}$. **P** contains the subgroup **S**, the direct sum of countably many copies of **Z**. In the literature, **P** is also denoted $\mathbf{Z}^{\mathbf{N}}$, or $\mathbf{Z}^{\omega}$. Since all the groups considered in this paper are abelian, it will save space to adopt the convention that the term group is short for abelian group. The textbooks [20] and [22] are good references for infinite abelian group theory. We use the symbols ω, ω_1, and 2^{ω} to denote the cardinal numbers of the natural numbers **N**, the first uncountable cardinal, and the cardinal number of the real numbers **R**, respectively. For example, **P** has cardinality 2^{ω}, and **S** has cardinality ω.

[1] I am very grateful to the organizers of the Irish Mathematical Society Conference in September 1997 for the opportunity to read a shorter version of this paper to the participants.

There are four sections in the paper. The first resumes some basic material on **P**; the second looks at a recently discovered connection relating slender subgroups of **P** to certain cardinal invariants of the real numbers. In the third section, some logical aspects of **P** are explored. The final section is about the complexity of the lattice of subgroups of **P**. Since the material covered in the paper is diverse and scattered across different domains, I have not given many proofs, hoping that the bibliography will enable the reader to follow themes in more depth.

1. Basics

The structure of infinite free groups is relatively clear (see, for example, [22]): for each infinite cardinal κ, there is exactly one free group of cardinality κ on κ generators (up to isomorphism). So an obvious initial question in the study of **P** is to determine how it stands in relation to freeness: is **P** free?

Definition A group G is **free** if G is (isomorphic to) a direct sum of copies of **Z**.

For example, **P** has a free subgroup **S**.

Theorem 1.1 (Baer [1]) *The group* **P** *is not free.*

We shall deduce this theorem from a stronger assertion below involving the notion of κ–freeness.

Definition Suppose that κ is an infinite cardinal. A group G is κ**–free** if every subgroup $H \leq G$ having less than κ elements is free.

Every free group is κ–free for every cardinal κ, since subgroups of free groups are free; if $\lambda < \kappa$, then κ–freeness implies λ–freeness. Questions about whether λ–freeness implies κ–freeness for $\lambda < \kappa$ are highly non-trivial and have stimulated one of the most important research orientations in infinite abelian group theory, leading for example to Shelah's singular compactness theorem [7, 25, 20] and independence results in set theory [30]. Since in general κ–freeness is weaker than freeness, we can refine the initial question about the non-freeness of **P** and ask whether there are any cardinals κ for which **P** is κ–free. Recall that ω_2 is the second uncountable cardinal, the cardinal successor of ω_1.

Proposition 1.2 *The group* **P** *is not* ω_2*–free.*
Proof: We need to find a non-free subgroup of **P** which has cardinality ω_1. Let p be a fixed prime number, and take a pure subgroup H of cardinality ω_1 containing **S** and such that every element $(n_1, n_2, \ldots)$ of H has the property that the tail is divisible by arbitrarily high powers of p: $(\forall m)(\exists r)(\forall k > r)(p^m | n_k)$.[2] Then the quotient group H/pH is a vector space over $\mathbf{F}_p$, the finite field of p elements. It follows that H is not free, for if H were free, then H/pH must have dimension (and hence cardinality) ω_1; but every coset of H/pH contains an element of **S**, and hence H/pH has cardinality at most $|\mathbf{S}| = \omega$ –a contradiction. So H is not free. ■

We can use Proposition 1.2 to improve exercise 19.7 of [22]:

Corollary *For every uncountable cardinal* κ*, there exists a non-free* ω_1*–free group of cardinality* κ*.*
Proof: For example, the group $\oplus_{\alpha<\kappa} H$, the direct sum of κ copies of the group H in the proof of Proposition 1.2, will work. ■

Proposition 1.2 implies Baer's theorem, since H is a non-free subgroup of **P**. However, it leaves open the question whether countable subgroups of **P** are free, i.e. whether **P** is ω_1–free.

Theorem 1.3 (Specker [42]) *The group* **P** *is* ω_1*–free.*
Proof: This is a well-known result and a full proof is given in the standard reference textbooks [20, 22]. It rests on Pontryagin's Criterion: a countable group is free if and only if every finite rank subgroup is free. We shall give a proof of this criterion using logic in section three. The proof of Theorem 1.3 proceeds as follows: every finite rank subgroup of **P** is embedded in a finitely generated torsion-free direct summand of **P**, and hence is free; so if $G \leq \mathbf{P}$ is countable, then every finite rank subgroup of G is free; now apply Pontryagin's Criterion. ■

To close this section, let us introduce another type of "local" freeness which has been intensively studied: a group G is almost free if G is $|G|$–free ($|G|$ is the number of elements of G), i.e. every

[2] Or: let H be an elementary submodel of the p-adic closure of **S** in **P** of cardinality ω_1 containing **S**. H exists by the Downward Loewenheim-Skolem Theorem of first-order logic.

subgroup of G of smaller cardinality than G is free. Is **P** almost free?

Corollary 1.4 **P** *is almost free if and only if the Continuum Hypothesis (*$\mathbf{CH}$*:* $2^\omega = \omega_1$*) holds.*

If the Continuum Hypothesis is true, then **P** is almost free if **P** is ω_1–free, which is true by Theorem 1.3; if the Continuum Hypothesis is false, then the subgroup H in Proposition 1.2 is a non-free subgroup of cardinality $\omega_1 < 2^\omega = |\mathbf{P}|$ and so **P** is not almost free. Thus one cannot decide whether **P** is almost free or not on the basis of ordinary set theory (ZFC).

One trend in the study of κ–freeness has been to try to find equivalences between the algebraic and set-theoretic definitions. In light of Corollary 1.4, it might be interesting to know whether there are algebraic properties φ and ψ such that:

(1) **P** has almost φ iff the weak Continuum Hypothesis (**wCH**: $2^\omega < 2^{\omega_1}$) holds;

(2) **P** has almost ψ iff Diamond holds.

Diamond $\diamond$ is a stronger form of the Continuum Hypothesis **CH**. One way to state **CH** is as a list of guesses A_α for the subsets of **N**:

$$(\exists \{A_\alpha \subseteq \alpha : \alpha < \omega_1\} \text{ such that}$$
$$(\forall X \subseteq \mathbf{N})(\{\alpha : \; X = A_\alpha\} \text{ is a stationary subset of } \omega_1)).$$

A stationary subset of ω_1 is large: it intersects every closed unbounded subset of ω_1 (in the order topology) non-trivially. In other words, the Continuum Hypothesis says that there is a list of length ω_1 which predicts correctly every subset of natural numbers a large number of times. What about subsets of ω_1? One cannot hope for a list of length ω_1 which would predict correctly every subset of ω_1, since there are 2^{ω_1} subsets of ω_1 and $2^{\omega_1} > \omega_1$. But perhaps one might be able to predict correctly just the initial segments of subsets of ω_1. Diamond asserts that there is a list of ω_1 guesses A_α for the initial segments of subsets of ω_1 and these guesses are correct on a large subset of ω_1. Formally, Diamond states:

$$(\exists\{A_\alpha \subseteq \alpha : \alpha < \omega_1\} \text{ such that}$$
$$(\forall X \subseteq \omega_1)(\{\alpha :\ X \cap \alpha = A_\alpha\} \text{ is a stationary subset of } \omega_1)).$$

A more detailed explanation of this type of prediction principle can be found in the standard textbooks on set theory [26] and [29].

2. P and the real numbers R

We start this section by looking at some cardinal invariants of the real numbers **R**. Recent research has uncovered rather surprising connections between these invariants and the size of certain subgroups of **P**.

Definition (1) The **additivity of measure**, **add(L)**, is the smallest number of measure-zero subsets of **R** whose union is not of measure zero.

(2) The **additivity of category**, **add(B)**, is the smallest number of first category subsets of **R** whose union is of second category (not of first category).

(3) The cardinal **d**, the **dominating number**, is defined as:

$$\min\{|D| :\ D \text{ is a subset of } \mathbf{N}^{\mathbf{N}} \text{ and}$$
$$(\forall g \in \mathbf{N}^{\mathbf{N}})(\exists f \in D)(g(n) \leq f(n) \text{ for all but finitely many } n)\}.$$

(4) The cardinal **b**, the **bounding number**, is defined as:

$$\min\{|B| :\ B \text{ is a subset of } \mathbf{N}^{\mathbf{N}} \text{ and}$$
$$(\forall g \in \mathbf{N}^{\mathbf{N}})(\exists f \in B)(g(n) < f(n) \text{ for all but finitely many } n)\}.$$

The countable additivity of Lebesgue measure and the Baire category theorem imply that **add(L)** and **add(B)** are both at least ω_1. And they are also at most 2^ω. It is immediate too that $\omega_1 \leq \mathbf{b} \leq \mathbf{d} \leq 2^\omega$.

The reason why the dominating and bounding numbers are called invariants of the reals is that the irrationals are homeomorphic to the topological space $\mathbf{N}^{\mathbf{N}}$ when $\mathbf{N}^{\mathbf{N}}$ is given the product topology and $\mathbf{N}$ has the discrete topology.

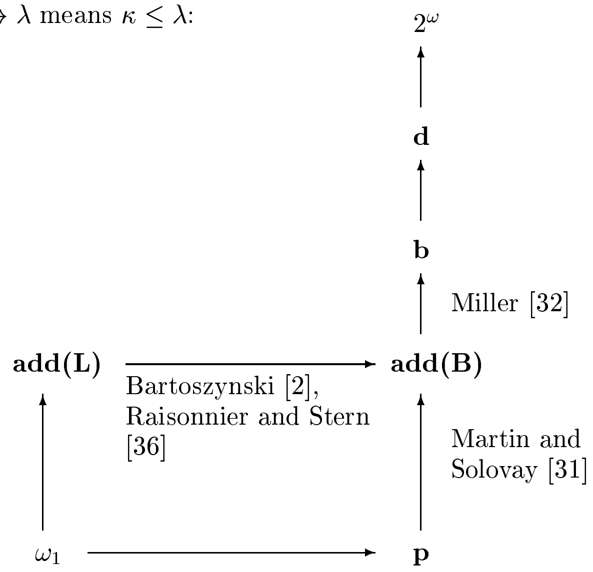
We shall need one other invariant which is called the pseudo-intersection number. A family F of subsets of $\mathbf{N}$ has the **strong finite intersection property** (SFIP) if the intersection of every finite subfamily is an infinite set. For example, the family of co-finite subsets of $\mathbf{N}$ has the SFIP. A set A is **almost contained** in a set B if $A \setminus B$ is finite.

Definition The cardinal $\mathbf{p}$ is

$$\min\{|F| : F \text{ is a family of subsets of } \mathbf{N} \text{ such that } F \text{ has the SFIP but there is no infinite set which is almost contained in every member of } F\}.$$

It is an instructive exercise to show that $\omega_1 \leq \mathbf{p} \leq 2^{\omega}$.

These cardinals are related as in the following picture, part of the Cichon diagram (except for the cardinal $\mathbf{p}$). An arrow relation $\kappa \to \lambda$ means $\kappa \leq \lambda$:



Some of the earliest results on the cardinal invariants of the reals are due to Rothberger [37]. The discovery of Martin's Axiom in 1970 stimulated renewed interest in these and other invariants. A fuller account of the area which has been studied in great depth by mathematicians since the 1970's is available in the articles by van Douwen [13] and Vaughan [44]. More recent work has revealed links between cardinal invariants and quadratic forms: several of these invariants determine how large orthogonal complements there are in a quadratic space. This research (on Gross and strongly Gross spaces) is surveyed in the paper [43]. The bounding number $\mathbf{b}$ also appears in recent work (in functional analysis) on metrizable barrelled spaces [38].

To explain the connection of cardinal invariants with the Baer-Specker group $\mathbf{P}$, we shall introduce one further definition.

Definition (Łoš). A group G is **slender** if whenever ϕ is a homomorphism from $\mathbf{P}$ into G, then $\phi(e_n) = 0$ for all but finitely many n, where e_n is the element of $\mathbf{P}$ that has 1 at the n–th co-ordinate and 0 everywhere else.

There are several important equivalent characterizations of slender groups which we insert here for the sake of completeness and as an aid to intuition.

Theorem 2.1 (Nunke [33], Heinlein [24], Eda (1982, see [20])) The group G is slender if and only if G does not contain a copy of the rationals $\mathbf{Q}$, the cyclic group of order p $\mathbf{Z}(\mathbf{p})$, the p-adic integers $\mathbf{J}_p$, or $\mathbf{P}$;
equivalently, every homomorphism ϕ from $\mathbf{P}$ into G is continuous, where G and $\mathbf{Z}$ are given the discrete topology and $\mathbf{P}$ the product topology;
equivalently, for any family $\{G_i : i \in I\}$ and homomorphism ϕ from $\Pi_{i\in I}G_i$ to G, there are ω_1-complete ultrafilters $D_1, \ldots, D_n$ on I such that

$$(\forall g \in \Pi_{i\in I}G_i)(\text{if the support of } g, \{i \in I : g(i) \neq 0\},$$
$$\text{does not belong to } D_k (1 \le k \le n), \text{ then } \phi(g) = 0).$$

Corollary 2.2 *Every ω_1–free group which does not contain $\mathbf{P}$ is slender.*

Examples The group **Z** is slender; every free group is slender. **P** is not slender (Specker [42]). Subgroups and direct sums of slender groups are slender.

Specker's proof that **P** is not slender works for many other subgroups of **P** which exhibit the **Specker phenomenon**. But these subgroups all have cardinality 2^ω.

Definition (Eda [15], Blass [8]) (1) A subgroup G of **P** exhibits the **Specker phenomenon** iff G contains a sequence $\{g_n : n \in \mathbf{N}\}$ of linearly independent elements such that whenever ϕ is a homomorphism from G into **Z**, then $\phi(g_n) = 0$ for all but finitely many n.

(2) The **Specker-Eda number, se**, is defined as:

$$\min\{|G| : G \leq \mathbf{P} \text{ exhibits the Specker phenomenon}\}.$$

Corollary 2.3 $\omega_1 \leq \mathbf{se} \leq 2^\omega$.

Theorem 2.4 (Eda [15].) (1) **CH** *implies* $\mathbf{se} = \omega_1$.

(2) *Martin's Axiom* (**MA**) *implies* $\mathbf{se} = 2^\omega$.

(3) *There is a model of ordinary set theory* (ZFC) *in which* $\mathbf{se} < 2^\omega$.

In 1994, Andreas Blass observed that Eda's proofs establish connections between the Specker-Eda number and some of the cardinal invariants of the real numbers which were defined at the beginning of this section.

Theorem 2.5 (Eda [15], Blass [8]) (1) $\mathbf{p} \leq \mathbf{se} \leq \mathbf{d}$.

(2) $\mathbf{add}(\mathbf{L}) \leq \mathbf{se} \leq \mathbf{b}$.

Conjecture 2.6 (Blass [8]) $\mathbf{se} = \mathbf{add}(\mathbf{B})$.

3. P and infinitary logic

From the logical point of view, a group G is a structure $\mathbf{G} = (G, +^G, -^G, 0^G)$ which satisfies the axioms for a group. In this section we use **L** to denote the vocabulary of groups, that is, **L** contains the constant, unary, and binary function symbols **0**, $-$, and $+$ to name the zero, inverse, and addition of a group. There are other possible choices for the vocabulary of groups, but for the sake of definiteness we shall fix **L** as above. It is also a fact that all the theorems of logic presented here are true in much greater generality.

The infinitary language $\mathbf{L}_{\infty\kappa}$ is the smallest class of formulas in the vocabulary **L** which is closed under negations, conjunctions of arbitrary length, and strings of quantifiers

$$(\exists x_1 \exists x_2 \ldots \exists x_\alpha \ldots)_{(\alpha<\lambda)}$$

of length λ less than κ. This infinitary language is more expressive than the first-order language of groups where one is limited to negations, finite conjunctions, and finite strings of quantifiers. While the axioms for a group are first-order, many of the interesting group-theoretic properties cannot be expressed by first-order sentences. For example, the following sentence of $\mathbf{L}_{\infty\omega}$ says that a group is torsion:

$$(\forall g)(g = 0 \text{ or } 2g = 0 \text{ or } 3g = 0 \text{ or } \ \ldots \text{ or } ng = 0 \ldots).$$

But the concept of torsion cannot be axiomatized in a first-order language. Infinitary logic can express the concepts of κ–freeness, κ–purity, $< \kappa$–generatedness and so on. The paper by Barwise [3] is an excellent introduction to the back and forth methods characteristic of infinitary logic. Other useful references for infinitary logics are the book by Barwise [4] and the article by Dickmann [12]. A typical problem in general infinitary model theory involves determining whether there are infinitarily equivalent non-isomorphic models in various cardinalities, [40].

Definition Two groups A and B are $\mathbf{L}_{\infty\kappa}$**–equivalent** if and only if for every sentence φ in $\mathbf{L}_{\infty\kappa}$: φ is true in A iff φ is true in B.

So $\mathbf{L}_{\infty\kappa}$–equivalent groups cannot be distinguished by a sentence in the infinitary language $\mathbf{L}_{\infty\kappa}$. There is an algebraic characterization of infinitary equivalence which is very useful and perhaps more familiar.

Theorem 3.1 (Karp [27], Benda [6], Calais [9]) *Two groups A and B are $\mathbf{L}_{\infty\kappa}$–equivalent if and only if there is a κ–**extendible system of partial isomorphisms** from A to B.*

A κ–extendible system of partial isomorphisms from A to B is a family F of isomorphisms between subgroups of A and B which has the κ–back-and-forth property: if $\phi \in F$ is an isomorphism from $M_1 \leq A$ onto $N_1 \leq B$ and X (respectively Y) is a subset of A (respectively B) of cardinality less than κ, then there exists $\psi \in F$ from M_2 onto N_2 such that $M_1 \leq M_2 \leq A$, $N_1 \leq N_2 \leq B$, ψ extends ϕ and X (Y) is a subset of M_2 (N_2).

Using this algebraic concept, it is easy to check for example that any two uncountable free groups are $\mathbf{L}_{\infty\omega}$–equivalent. κ–extendible systems are a natural generalization of Cantor's technique for showing that there is (up to isomorphism) exactly one unbounded dense countable linear order, namely the linearly ordered set of the rational numbers [3]. Indeed, for countable groups, infinitary equivalence and isomorphism are synonymous:

Theorem 3.2 (Scott [39]) *If A and B are countable $\mathbf{L}_{\infty\omega}$–equivalent groups, then A and B are isomorphic.*

The infinitary model theory of abelian groups was intensively studied in the 1970's by Barwise, Eklof, Fischer, Gregory, Kueker, and Mekler (see [5, 16, 17, 23, 28] for example). One of the first important results is due to Eklof, who succeeded in determining which groups are infinitarily equivalent to free groups.

Definition A subgroup A of G is κ–pure if for every subgroup B such that $A \leq B \leq G$ and B/A is $< \kappa$–generated (i.e. generated by fewer than κ elements), A is a direct summand of B.

Theorem 3.3 (Eklof [16]) *A group G is $\mathbf{L}_{\infty\kappa}$–equivalent to a free group if and only if every $< \kappa$–generated subgroup of G is contained in a free, κ–pure subgroup of G.*

For the purposes of this exposition, it is sufficient to know the following corollaries.

Corollary 3.4 *A group G is $\mathbf{L}_{\infty\omega}$–equivalent to a free group iff every subgroup of G of finite rank is free.*

Eklof used this result to deduce a very famous criterion for freeness in countable groups:

Corollary 3.5 (Pontryagin's Criterion [35]) *A countable group is free iff every subgroup of finite rank is free.*

Proof: Apply Scott's Theorem 3.2 to Corollary 3.4. ■

Corollary 3.6 (Kueker) *A group is $\mathbf{L}_{\infty\omega}$–equivalent to a free group iff it is ω_1–free.*

Now we can return to the Baer-Specker group **P** and see what these facts tell us.

Corollary 3.7 (Keisler-Kueker) *The Baer-Specker group* **P** *is $\mathbf{L}_{\infty\omega}$–equivalent to a free group. The class of free groups is not definable in $\mathbf{L}_{\infty\omega}$.*

Corollary 3.8 (Eklof [16]) *The group* **P** *is not $\mathbf{L}_{\infty\omega_1}$–equivalent to a free group.*

Since free groups are slender, it follows too from Corollary 3.7 that the class of slender groups is not definable in $\mathbf{L}_{\infty\omega}$. It might be tempting to conjecture that **P** is not $\mathbf{L}_{\infty\omega_1}$–equivalent to a slender group. Another possible suggestion is that **P** is not $\mathbf{L}_{\infty\mathbf{se}}$–equivalent to a slender group. Mekler showed that if κ is a strongly compact cardinal, then the class of free groups is definable in $\mathbf{L}_{\infty\kappa}$. This prompts the question whether the class of slender groups is definable in $\mathbf{L}_{\infty\kappa}$ if κ is strongly compact. Eklof and Mekler have developed applications of other generalized logics to problems of abelian group theory in the papers [18] and [19].

4. The lattice of subgroups of P

The broad thrust in this section is to describe some recent research on the complexity of the lattice of subgroups of **P**. One way to measure this complexity is to study what sorts of groups can be embedded into **P**. Very generally, the natural questions often have the form whether there are families of maximal possible size of subgroups of **P** which are strongly different (non-isomorphic) in some precisely defined sense.

An example of this type of theorem in the context of general abelian groups is the following.

Theorem 4.1 (Eklof, Mekler and Shelah [21]) *Under various set-theoretic hypotheses, there exist families of maximal possible size of almost free abelian groups which are pairwise almost disjoint (the intersection of any pair contains no non-free subgroup).*

There is an inverse correlation between the size of the family and the strong difference of its members. If one considers families of pure subgroups of **P** and takes the notion of strong difference to mean that the only homomorphisms between any pair are those of finite rank, then it is possible to prove the existence of a strongly different family of maximal size.

Theorem 4.2 (Corner and Goldsmith [10]) *Let* **D** *be the subgroup of* **P** *containing* **S** *such that* **D**/**S** *is the divisible part of* **P**/**S**. *Let* $c = 2^\omega$. *There exists a family* **C** *consisting of* 2^c *pure subgroups* G *of* **D** *with* $\mathbf{D}/G$ *rank–1 divisible where each* G *is slender, essentially-indecomposable, essentially-rigid with* $\mathrm{End}(G) = \mathbf{Z} + E_0(G)$, *where* $E_0(G)$ *is the ideal of all endomorphisms of* G *whose images have finite rank.*

A similar type of question is the following: does there exist a family of 2^{ω_1} non-isomorphic pure subgroups of **P**, each of cardinality ω_1, such that the intersection of any pair is free? The answer is positive.

Theorem 4.3 (Shelah and Kolman [41]) *There exists a family* $\{G_\alpha : \alpha < 2^{\omega_1}\}$ *of pure subgroups of* **P** *such that*
(1) *each* G_α *has cardinality* ω_1;
(2) *if* $\alpha \neq \beta$, *then* $G_\alpha \cap G_\beta$ *is free.*

The question whether certain classes of group can be embedded in **P** sometimes leads to independence results. Recall that a group G is ω_1–separable if every countable subset of G is contained in a free direct summand of G.

Theorem 4.4 (Dugas and Irwin [14]) *Embeddability of* ω_1–*separable groups of cardinality* ω_1 *in* **P** *is independent of ZFC.*

Reflexive subgroups of **P** have also been studied in some depth. The following result was known for many years under the additional assumption of the Continuum Hypothesis.

Theorem 4.5 (Ohta [34]) *There exists a non-reflexive dual subgroup of* **P**.

The variety displayed in this selection of results on the Baer-Specker group **P** illustrates how this easily definable abelian group is a source of interesting research problems whose solutions often reveal unexpected connections with problems in other domains of mathematics.

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