## Book Review

### Mathematics—The Music of Reason

Translated from the French by J. Dales and H. G. Dales

J. Dieudonné

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**Reviewed by** Robin Harte

Here is a master of exposition at the peak of his form - the interpreter and expositor of Grothendieck's theories offers us a dancing run over the surface of modern mathematics, carrying us from astronomy in the ancient world to Gödel, "independence" and Cohen "forcing". As we might expect, the perspective throughout is very "Bourbaki". After two more or less introductory chapters on "Mathematics and Mathematicians" and "The Nature of Mathematical Problems", each chapter is addressed to non-specialists and then furnished with an Appendix for the professionals. Thus we have "Objects and Methods in Classical Mathematics", with an Appendix ranging from ratios à la Euclid to limits via exhaustion, "Some Problems of Classical Mathematics" with an Appendix covering prime numbers and the Riemann zeta function, "New Objects and New Methods", whose Appendix is about Galois Theory and the foundations of metric spaces, and finally "Problems and Pseudo-problems about Foundations", with an Appendix about surface geometry and models of the real numbers. The translation, by J. and H. G. Dales, is uniformly excellent: only the typeface seems to have an old fashioned air about it.

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## Solutions to the Problems of the 36th IMO

**1. First solution.** Let $DN$ meet $XY$ at the point $R$. The triangles $RZD$ and $BZP$ are similar and hence $RZ/ZD = BZ/ZP$. Thus $RZ = BZ.ZD/ZP = ZX^2/ZP$. If $S$ is the point of intersection of $AM$ and $XY$, then a similar argument proves that $SZ = ZX^2/ZP$. Thus the points $R$ and $S$ coincide and the result follows.

**Second solution.** Choose coordinates so that the line $ABCD$ is the $x$-axis with $Z$ as origin and $XY$ is the $y$-axis. Let the coordinates of $A$, $B$, $C$, $D$ and $P$ be $(a,0)$, $(b,0)$, $(c,0)$, $(d,0)$ and $(0,p)$, respectively. The problem can now be solved using routine calculations.

**2.** The expression on the left hand side of the inequality can be made a little more friendly by letting $a = 1/x, = 1/y$ and $c = 1/z$. Then $xyz = 1$ and the inequality to be proved is:

$$\frac{x^2}{y+z} + \frac{y^2}{z+x} + \frac{z^2}{x+y} \geq \frac{3}{2}.$$

If $S$ denotes the left hand side, then

$$2(x+y+z)S = [(x+y)+(y+z)+(z+x)]S =$$
$$[(\sqrt{x+y})^2 + (\sqrt{y+z})^2 + (\sqrt{z+x})^2]\times$$
$$[(\frac{z}{\sqrt{x+y}})^2 + (\frac{x}{\sqrt{y+z}})^2 + (\frac{y}{\sqrt{z+x}})^2] \geq (z+x+y)^2$$

by Cauchy's inequality. But the arithmetic-geometric mean inequality gives $x+y+z \geq 3$, since $xyz = 1$. Thus

$$2(x+y+z)S \geq 3(x+y+z).$$

Hence $S \geq 3/2$ and the result is proved.

**3.** If $A_1$, $A_2$, $A_3$, $A_4$ are the vertices of a square of unit area and if $r_i = 1/6$ for $i = 1,2,3,4$, then the triangle $A_iA_jA_k$ has area $r_i + r_j + r_k$ for each triple $i$, $j$, $k$ $(1 \leq i < j < k \leq 4)$. So the result holds for $n = 4$.

Suppose that $B_1B_2B_3B_4$ is a convex quadrilateral and that there exist real numbers $r_1$, $r_2$, $r_3$, $r_4$ such that the area of the triangle $B_iB_jB_k$ is $r_i + r_j + r_k$ for all integers $i$, $j$, $k$ with $1 \leq i < j < k \leq 4$. Let $B$ be the point of intersection of $B_1B_3$ and $B_2B_4$. Then, by considering the areas of the triangles $BB_3B_4$, $BB_2B_3$, $B_1B_2B$, $B_1BB_4$, it is not difficult to prove that

$$(*) \qquad r_1 + r_3 = r_2 + r_4.$$

Let $n \geq 5$ and suppose there exist $n$ points $A_1, A_2, \ldots, A_n$ and $n$ real numbers $r_1, r_2, \ldots, r_n$ satisfying the conditions of the problem. Form the smallest convex set $C$ containing all the points $A_1$, $A_2$, $\ldots$, $A_5$.

**Case 1.** Suppose $C$ is a convex pentagon. Apply $(*)$ to the convex quadrilaterals $A_1A_2A_3A_4$ and $A_1A_2A_3A_5$ to get

$$r_1 + r_3 = r_2 + r_4 \text{ and } r_1 + r_3 = r_2 + r_5.$$

Thus $r_4 = r_5$. Repeating this argument for the appropriate pairs of quadrilaterals we get

$$r_1 = r_2 = r_3 = r_4 = r_5 = r,$$

say. Then the area of the triangle $A_2A_3A_4 = 3r =$ the area of the triangle $A_2A_3A_5$ and thus $A_2A_3$ is parallel to $A_4A_5$. Also, the area of $A_1A_2A_3 =$ the area of $A_2A_3A_5$ and hence $A_2A_3$ is parallel to $A_1A_5$. Thus $A_1$, $A_4$ and $A_5$ are collinear. This contradiction proves that $C$ is not a convex pentagon.

**Case 2.** Suppose $C$ is a convex quadrilateral. Without loss of generality, let $C$ be $A_1A_2A_3A_4$. Since no three of the points $A_1$, $A_2$, $\ldots$, $A_5$ are collinear we may also suppose, without loss of generality, that $A_5$ is strictly inside the triangle $A_1A_2A_4$. The equation

$$\begin{aligned}&\text{area } A_1A_2A_4 + \text{ area } A_2A_3A_4 = \\ &\text{area } A_2A_3A_5 + \text{ area } A_3A_4A_5 + \\ &\text{area } A_4A_1A_5 + \text{ area } A_1A_2A_5\end{aligned}$$



implies that $r_1 + r_3 + 4r_5 = 0$ and, hence, $r_2 + r_4 + 4r_5 = 0$ (using $(*)$). The equation

$$\text{area } A_1A_2A_4 = \text{ area } A_1A_2A_5 + \text{ area } A_1A_4A_5 + \text{ area } A_2A_4A_5$$

implies that $r_1 + r_2 + r_4 + 3r_5 = 0$. Thus $r_1 = r_5$. Thus the triangles $A_2A_3A_5$ and $A_2A_3A_1$ have the same area. Hence the triangle $A_1A_2A_5$ has zero area. This implies that $A_1$, $A_2$ and $A_5$ are collinear, which is impossible. Hence $C$ is not a quadrilateral.

**Case 3.** Suppose $C$ is a triangle. Then, without loss of generality $C = A_1A_2A_3$. The points $A_4$ and $A_5$ are inside the triangle. The equation

$$\text{area } A_1A_2A_4 + \text{ area } A_2A_3A_4 + \text{ area } A_3A_1A_4 = \text{ area } A_1A_2A_3$$

implies that $r_1 + r_2 + r_3 + 3r_4 = 0$. Similarly, on replacing $A_4$ by $A_5$, we get $r_1 + r_2 + r_3 + 3r_5 = 0$. Thus $r_4 = r_5$. Then

$$r_1 + r_2 + r_4 = r_1 + r_2 + r_5$$

implies that $A_4A_5$ is parallel to $A_1A_2$ and

$$r_1 + r_3 + r_4 = r_1 + r_3 + r_5$$

implies that $A_4A_5$ is parallel to $A_1A_3$. Thus $A_1$, $A_2$, $A_3$ are collinear. This contradiction implies that $C$ is not a triangle.

Since we get a contradiction in all cases we must have $n < 5$. Hence $n = 4$ is the only integer greater than 3 satisfying the conditions of the problem.

**4.** It is easy to deduce from condition (ii) that, for each integer $i \geq 1$,

$$\text{either } x_i = \frac{1}{2}x_{i-1} \text{ or } x_i = \frac{1}{x_{i-1}}.$$

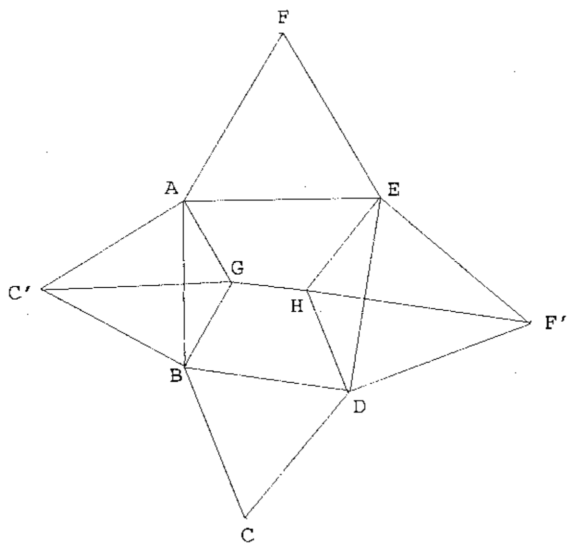
For each integer $i \geq 1$, induction can be used to prove that $x_i = 2^r x_0^s$, for some integer $r$ with $-i \leq r < i$ and $s = (-1)^{i-t}$, where $t = |r|$.

Let $x_{1995} = 2^r x_0^s$. Then $x_0 = x_{1995}$ gives $x_0^{1-s} = 2^r$. If $s = 1$ then $r = 0$. But this gives the contradiction $1 = s = (-1)^{1995-0} = -1$. Hence $s = -1$ and $x_0^2 = 2^r$. So the largest value $x_0$ can have is attained when $r$ here has its largest possible value. Now $-1995 \leq r < 1995$. The value $r = 1994$ is attained for the sequence which satisfies

$$x_{i+1} = \frac{1}{2}x_i, \text{ for } i = 0, 1, \ldots, 1993 \text{ and } x_{1995} = \frac{1}{x_{1994}}.$$

Then $x_{1995} = 2^{1994}x_0^{-1}$. So, in this case, $x_0^2 = 2^{1994}$. Thus $x_0 = 2^{997}$ is the maximum value of $x_0$ for which a sequence with the required properties exists.

**5.**




Form the quadrilateral $ABDE$. The triangles $BCD$ and $FAE$ are clearly equilateral. On the line segment $AB$ construct the (exterior) equilateral triangle $AC'B$ and on the line segment $DE$



construct the (exterior) equilateral triangle $EDF'$. In the quadrilaterals $CBAF$ and $C'BDF'$ we have $CB = C'B$, $BA = BD$ and $AF = DF'$. Also $\angle CBA = \angle C'BD$ and $\angle BAF = \angle BDF'$ because $\angle BAE = \angle BDE$, since the triangles $ABD$ and $AED$ are isosceles. Thus the quadrilaterals $CBAF$ and $C'BDF'$ are congruent. So $CF = C'F'$.

Since $\angle AC'B = 60°$ and $\angle AGB = 120°$, the quadrilateral $AC'BG$ is cyclic. Ptolemy's theorem then says that

$$AC'.BG + BC'.AG = AB.C'G$$

and thus,

$$BG + GA = GC',$$

since the triangle $AC'B$ is equilateral.
Similarly

$$EH + HD = HF'.$$

Thus

$$AG + GB + GH + HD + HE = C'G + GH + HF' \geq C'F' = CF,$$

and the result is proved.

**Note.** The result is still true without the condition $\angle AGB = \angle DHE = 120°$. This follows from the fact that the extended version of Ptolemy's theorem (applied to the quadrilateral $AC'BG$):

$$AC'.BG + BC'.AG \geq AB.C'G$$

applies when the angle $\angle AGB$ is arbitrary ($< 180°$). The inequalities

$$BG + GA \geq C'G \text{ and } DH + HE \geq HF'$$

can then be used to prove the result.

**6. First solution.** Set $\Omega = \{1, 2, \ldots, 2p\}$ and let $S$ be the collection of all the subsets of $\Omega$ each of which contains $p$ elements. Then $S$ contains $\binom{2p}{p}$ sets.

For each set $X \in S$ let $s(X)$ denote the sum of the elements of $X$. Let $B = \{1, 2, \ldots, p\}$ and $C = \{p+1, p+2, \ldots, 2p\}$. Then $B, C \in S$ and $s(B) \equiv s(C) \equiv 0 \bmod p$. If $A \in S$ and $A \neq B, C$ then $A \cap B \neq \emptyset$ and $A \cap C \neq \emptyset$. Let $T$ be the collection of sets obtained by excluding $B$ and $C$ from $S$. Then $T$ contains $\binom{2p}{p} - 2$ sets. Partition $T$ into collections of sets as follows: two sets $A$ and $A'$ are in the same collection if and only if $A \cap C = A' \cap C$ and there exists an integer $m$ with $0 \leq m < p$ such that

$$A' \cap B = \{x + m \bmod p : x \in A \cap B\}.$$

Then each such collection contains $p$ sets. Let $A$ and $A'$ be distinct sets in the same collection and suppose $A \cap B$ has $n$ elements. Then $0 < n < p$ and there exists an integer $m$ with $0 < m < p$ such that

$$A' \cap B = X \cup Y,$$

where

$$X = \{x + m : x \in A \cap B, x + m \leq p\}$$

and

$$Y = \{x + m - p : x \in A \cap B, x + m > p\}.$$

Then $s(A') - s(A) \equiv mn \bmod p$. But $p$ does not divide $mn$. Thus, if we calculate $s(A) \bmod p$ for each of the $p$ sets in any of the collections, we get all of the residues $0, 1, \ldots, p-1$. In particular, each collection contains exactly one set $A$ satisfying $s(A) \equiv 0 \bmod p$. Hence $T$ contains

$$\frac{1}{p}\left(\binom{2p}{p} - 2\right)$$

sets such that the sum of the elements in each set is divisible by $p$. Hence the number of $p$-element subsets of $\Omega$ such that the sum of the elements in each subset is divisible by $p$ is

$$\frac{1}{p}\left(\binom{2p}{p} - 2\right) + 2.$$



**Second Solution.** Set $\Omega = \{1, 2, \ldots, 2p\}$ as before and let $n_j$ denote the number of $p$-element subsets of $\Omega$ such that the sum of the elements of each subset is congruent to $j \bmod p$, for $j = 0, 1, \ldots, p-1$. Form the generating function

$$f(x) = \sum_{0}^{p-1} n_j x^j$$

of the sequence $n_0, n_1, \ldots, n_{p-1}$. Let $\omega$ be a primitive $p$-th root of 1. If $A = \{i_1, i_2, \ldots, i_p\}$ is a $p$-subset of $\Omega$ such that

$$i_1 + i_2 + \ldots + i_p \equiv j \bmod p,$$

then

$$\omega^{i_1 + i_2 + \ldots + i_p} = \omega^j.$$

Thus

$$f(\omega) = \sum \omega^{i_1 + i_2 + \ldots + i_p},$$

where the sum is taken over all $p$-subsets $A$ of $\Omega$ as above. The coefficient of $x^p$ in the product

$$(x - \omega)(x - \omega^2) \cdots (x - \omega^{2p})$$

is

$$(-1)^p \sum \omega^{i_1 + i_2 + \ldots + i_p} = -f(\omega).$$

But the product equals

$$\{(x - \omega)(x - \omega^2) \ldots (x - \omega^p)\}^2 = (x^p - 1)^2 = x^{2p} - 2x^p + 1.$$

Thus $f(\omega) = 2$. So

$$n_0 - 2 + n_1\omega + n_2\omega^2 + \ldots + n_{p-1}\omega^{p-1} = 0.$$

But $\omega$ is any primitive $p$-th root of 1. So, if

$$g(x) = n_0 - 2 + n_1 x + n_2 x^2 + \cdots + n_{p-1} x^{p-1},$$

then

$$g(\omega) = g(\omega^2) = \cdots = g(\omega^{p-1}) = 0,$$

because $\omega$, $\omega^2$, ..., $\omega^{p-1}$ are all the primitive $p$-th roots of 1. Thus

$$g(x) = (x-\omega)(x-\omega^2)\cdots(x-\omega^{p-1})h(x),$$

for some polynomial $h(x)$. By comparing degrees we see that $h(x) = k$, a constant. Thus

$$g(x) = k(1 + x + x^2 + \cdots + x^{p-1}).$$

Thus

$$n_0 - 2 = n_1 = \cdots = n_{p-1} = k.$$

But

$$n_0 + n_1 + \cdots + n_{p-1} = \binom{2p}{p}.$$

Hence

$$n_j = \frac{1}{p}\left(\binom{2p}{p} - 2\right)$$

for $j = 1, 2, \ldots, p-1$ and

$$n_0 = \frac{1}{p}\left(\binom{2p}{p} - 2\right) + 2.$$

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