companion to it. Also there is a very sparse amount of material on procedures in Maple. Overall this book is a very welcome addition to the literature on Maple.

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## Book Review

### Theory of Singular Boundary Value Problems

D. O'Regan
World Scientific, Singapore, 1994, xi+154pp
ISBN 981-02-1760-9
Price $ 38.00, hardback.

**Reviewed by** Johnny Henderson

The last decade has given rise to much activity in the area of boundary value problems (BVP's) for singular ordinary differential equations (ODE's), with this book's author contributing significantly to that activity. The book under review presents some topics of current interest in the theory of regular and singular BVP's (singular in both independent and dependent variables), with the two objectives to serve as a graduate text on the existence theory for these problems, as well as acquainting researchers new to the field with results and methods. The author states that no attempt has been made to deal in greatest generalities, and yet while the book is restricted to second order ODE's, a very general theory is developed for singular two-point BVP's in this context. While the book is self-contained, a reasonable background in real and functional analysis is assumed on the part of the reader.

There are ten clearly written chapters. While there are no formally listed exercises, the work involved in verifying results for cases analogous to those the author presents in detail serves as an adequate set of exercises. References are included at the end of each chapter.

Chapter 1 is an introduction, which serves as motivation for the study of singular two-point BVP's for second order ODE's, via presentation of problems involving, for example, the study of steady-state oxygen diffusion in a cell with Michaelis-Menten kinetics, the determination of the electrical potential in an atom due to Thomas and Fermi, and the study of the Emden-Fowler equation for the non-linear phenomena in non-Newtonian fluid theory. Chapter 2 is devoted to Fixed Point Theory which the author primarily will use in establishing solutions of regular and singular BVP's including the problem mentioned in the first chapter. More precisely, the author develops in detail a non-linear alternative theory known as the Leray-Schauder Alternative, based on essential mappings and homotopy equivalence within the framework of topological transversality which A. Granas introduced in 1976. Many of the book's existence results rely on the application of the following result.

**Theorem (Non-linear Alternative).** *Let $C$ be a convex subset of a normed linear space $E$ and let $U$ be an open subset of $C$ with $p^* \in U$. Let $F : \bar{U} \to C$ be a compact continuous map. Then at least one of the following holds:*

*(i) $F$ has a fixed point.*

*(ii) There is an $x \in \partial U$ with $x = \lambda F(x) + (1-\lambda)p^*$, for some $\lambda$ with $0 < \lambda < 1$.*

Application of this theorem is first made in Chapter 3 in obtaining solutions in Section One of the equation

$$\frac{1}{p}(py')' = qf(t,y,py'), \quad 0 < t < 1, \tag{1}$$

and in Section Two of the equation

$$\frac{1}{p}(py')' + ry + \kappa py' = f(t,y,py'), \quad \text{a.e. on } [0,1], \tag{2}$$

satisfying boundary conditions of various types:

$$\text{(Sturm-Liouville)} \begin{cases} -\bar{\alpha}y(0) + \bar{\beta}\lim_{t\to 0^+} p(t)y'(t) = c_0, \\ ay(1) + b\lim_{t\to 1^-} p(t)y'(t) = c_1, \end{cases} \tag{3}$$

where $\bar{\alpha} > 0$, $\bar{\beta} \geq 0$ in the first equation and $a \geq 0$, $b \geq 0$, $a^2 + b^2 > 0$ in the second.

$$\text{(Mixed)} \begin{cases} \lim_{t\to 0^+} p(t)y'(t) = c_0, \\ ay(1) + b\lim_{t\to 1^-} p(t)y'(t) = c_1, \quad a > 0, b \geq 0, \end{cases} \tag{4}$$



$$\text{(Neumann)} \begin{cases} \lim_{t\to 0^+} p(t)y'(t) = c_0, \\ \lim_{t\to 1^-} p(t)y'(t) = c_1, \end{cases} \tag{5}$$

$$\text{(Periodic)} \begin{cases} y(0) = y(1), \\ \lim_{t\to 0^+} p(t)y'(t) = \lim_{t\to 1^-} p(t)y'(t), \end{cases} \tag{6}$$

$$\text{(Bohr)} \begin{cases} y(0) = c_0, \\ \int_0^1 \frac{ds}{p(s)} \lim_{t\to 1^-} p(t)y'(t) - y(1) = c_1, \end{cases} \tag{7}$$

where in the case of (1) with (3) and (1) with (4), $f : [0,1] \times \mathbf{R}^2 \to \mathbf{R}$ is continuous, $q \in C(0,1)$, $p \in C[0,1] \cap C^{(1)}(0,1)$ and both $p > 0$ and $q > 0$ on $(0,1)$, while in the case of (2) with any of (3) to (7), assumptions include $pf : [0,1] \times \mathbf{R}^2 \to \mathbf{R}$ is $L^1$-Caratheodory, and $p\kappa \in L^1[0,1]$. To apply the Non-linear Alternative to obtain solutions of, say, (1) with (3), *a priori* bounds, independent of parameter $\lambda$, are exhibited on solutions of an associated family of problems,

$$\frac{1}{p}(py')' = \lambda qf(t,y,py'), \quad 0 < t < 1, \quad 0 < \lambda < 1, \tag{8}$$

satisfying condition (3). The first existence theorem, Theorem 3.3, states that if this *a priori* bound exists on solutions of (8) with condition (3), for all $\lambda$, and if certain integrability is assumed,

$$\int_0^1 \frac{ds}{p(s)} < \infty \text{ and } \int_0^1 p(s)q(s)ds < \infty,$$

then (1) with condition (3) has a solution. The *a priori* bound arguments employed by the author involve tremendous amounts of calculations, often tedious, yet these arguments give an excellent display of the work required to obtain bounds necessary to apply the Alternative. As such, the arguments are rather elegant. The statement of the first existence theorem is typical of most throughout the book and the methods set the tone for the arguments to be employed. For example, in dealing with the singular problems (2) with conditions (3)-(7), it is also necessary to assume that the corresponding homogeneous BVP has only the trivial solution, in effect, giving rise to a Green's function, which is then used to define a compact mapping $F$ to which the Non-linear Alternative is applied.

The purpose of Chapter 4 is to apply operator theory methods to obtain solutions of regular and singular eigenvalue problems

$$Ly = \lambda y, \quad 0 < t < 1, \tag{9}$$

satisfying any of the homogeneous boundary conditions corresponding to (3)-(7), where

$$Ly = \frac{1}{pq}(py')'.$$

In addition to the hypotheses above on $p$ and $q$ in the case of (1) and condition (3), it is assumed that the domain of $L$, $D(L)$, is given by

$$\begin{aligned} D(L) = \{v \in C[0,1] : v, pv' \in AC[0,1],\ (pv')' \in L^2[0,1], \\ \text{and } -\bar{\alpha}v(0) + \bar{\beta} \lim_{t\to 0+} p(t)v'(t) \\ = av(1) + b \lim_{t\to 1^-} p(t)v'(t) = 0\}. \end{aligned}$$

The operator

$$L^{-1} : L^2_{pq}[0,1] \to D(L) \subseteq L^2_{pq}[0,1]$$

is a completely continuous, symmetric operator (making use of the Green's function), by which operator theory results yield an infinite number of real eigenvalues of $L$ with corresponding eigenvectors in $D(L)$, as well as establishing some Rayleigh-Ritz integral inequalities, such as the Wirtinger inequality when $\beta = b = 0$ and $p = q = 1$. Similar results are obtained for the cases of mixed, Neumann, periodic and Bohr problems with each involving appropriate $D(L)$. Again the mechanics could be described as tedious, yet they are beautifully done. Within the chapter's development, a well-written and detailed exposition is given on the spectrum of a symmetric, completely continuous operator.



The first part of Chapter 5 is devoted to upper and lower solutions methods for obtaining solutions of (1) with condition (3) (where $a > 0$ and $q$ is assumed positive on $(0,1)$ such that $pq \in L^1[0,1]$, with the same hypotheses on $p$ and $f$ as when (1) with condition (3) was treated in Chapter 3). An upper solution $\beta$ is defined in the natural way to mean $\beta \in C[0,1] \cap C^{(2)}(0,1)$ such that $p\beta' \in C[0,1]$ and satisfies

$$qf(t,\beta,p\beta') \geq \frac{1}{p}(p\beta')', \quad 0 < t < 1,$$

$$\bar{\alpha}\beta(0) + \bar{\beta} \lim_{t\to 0^+} p(t)\beta'(t) \leq c_0,$$
$$a\beta(1) + b \lim_{t\to 1^-} p(t)\beta'(t) \geq c_1,$$

with a lower solution $\alpha$ defined by reversing the inequalities. The results are such that, if there exist upper and lower solutions $\beta$ and $\alpha$ of (1) with $\alpha \leq \beta$ on $[0,1]$ in conjunction with a Nagumo type condition for the singular setting, along with additional technical hypotheses, then (1), (3) has a solution $y \in C[0,1] \cap C^{(2)}(0,1)$ with $\alpha \leq y \leq \beta$ and $py' \in C[0,1]$. The chapter is fairly complete, including a discussion of radial solutions of elliptic PDE's in spherical domains for which $\frac{1}{p} \notin L^1[0,1]$, since $p(t) = t^{n-1}$. Yet exchanging that integrability condition for

$$\int_0^1 \frac{1}{p(s)} \int_0^s p(x)q(x)dx\, ds < \infty,$$

along with some of the technical hypotheses, upper and lower solutions methods are successfully applied. The chapter then returns to applications of the Non-linear Alternative, when, motivated by the Thomas-Fermi equation, *a priori* bounds are established for solutions of the associated one-parameter family

$$\frac{1}{p}(py')' = \lambda q f(t,y), \quad 0 < \lambda < 1, \quad 0 < t < b,$$

satisfying

$$y(0) = a, \quad p(b) \int_0^b \frac{ds}{p(s)} y'(b) - y(b) = 0.$$

Chapter 6 first makes use of Rayleigh-Ritz minimization theorems with respect to the $L^2_{pq}$-norm to establish existence of solutions of (1) with (3), with $c_0 = c_1 = 0$, where $f$ can be decomposed as

$$f(t, u, v) = g(t, u, v) + h(t, u, v),$$

with $g, h : [0, 1] \times \mathbf{R}^2 \to \mathbf{R}$ both continuous,

$$|h(t, u, v)| \leq K\{|u|^\gamma + |v|^\tau + 1\}, \quad 0 \leq \gamma,\ \tau < 1,$$

and for $C \in \mathbf{R}$ and $d \leq 0$,

$$ug(t, u, v) \geq C|u|^2 + d|uv|,$$

and

$$|g(t, u, v)| \leq A(t, u)|v|^2 + B(t, u),$$

where $A$ and $B$ are bounded on bounded sets. Also $p(t)\sqrt{q(t)}$ is bounded on $[0, 1]$. Then very nice applications of Hölder's inequality and the results of Chapter 4 yield *a priori* bounds on solutions of corresponding one-parameter family of problems, so that the Non-linear Alternative can be applied to yield solutions of (1) with (3) (with $c_0 = c_1 = 0$), provided

$$C^+ - dN_0\sqrt{\mu} < \mu,$$

where $\mu$ is the first eigenvalue of $Ly = \lambda y$ satisfying condition (3) (with $c_0 = c_1 = 0$ and $L$ is as in Chapter 4), $C^+ = \max\{0, -C\}$, $N_0 = \sup_{[0,1]} p(t)\sqrt{q(t)}$. Similar treatment is given to the case when $p\sqrt{q}$ is singular at $t = 0$ and/or $t = 1$, even including

$$\int_0^1 \frac{ds}{p(s)} = +\infty.$$

The chapter concludes with a discussion of the non-existence and existence of solutions of (1) satisfying, for example, $y(0) = y(1) = 0$. With the equations

$$y'' = (y')^2 + \pi \text{ and } y'' = (y')^2 - \pi^2$$



as models, the author points out that what is important is not the growth of solutions, as $|y'| \to \infty$, but is rather the zero set of $f$. Once again for the case of existence, *a priori* bounds on solutions of an associated one-parameter family are exploited, leading to an application of the Non-linear Alternative.

In Chapter 7, the author considers singular BVP's for

$$\frac{1}{p}(py')' + \mu qy = f(t, y, y') \text{ a.e. on } [0, 1], \tag{10}$$

for the non-resonant case $\lambda_{m-1} < \mu < \lambda_m$, and for the resonant case $\mu = \lambda_m$, $m = 1, 2, \ldots$, where $\lambda_0 = -\infty$ and the $\lambda_i$ are assumed to be eigenvalues of the appropriate homogeneous problem associated with Sturm-Liouville, Neumann or periodic boundary conditions, and where $f$ decomposes as

$$f(t, u, v) = \eta v + g(t, u, v),$$

with $\eta p \in L^1[0, 1]$, $pq \in L^1$-Caratheodory, and

$$|g(t, u, v)| \leq \varphi_1(t) + \varphi_2(t)|u|^\gamma + \varphi_3(t)|v|^\theta,$$

$p\varphi_i \in L^1[0, 1]$, and

$$\sup_{[0,1]} \int_0^1 |p(t)G(t, s)\eta(s)| ds < 1$$

($G(t, s)$ is the Green's function for the respective BVP). In the case of non-resonance, *a priori* bounds are established again for solutions of an associated one-parameter family of BVP's, so that the Non-linear Alternative can be applied. For the case of resonance, two types of existence results are presented: the first is for singular problems on the "left" of the eigenvalue and the second is for singular problems on the "right" of the eigenvalue. The arguments give much insight of the work required to obtain the necessary *a priori* bounds on solutions, and in these cases exhibit nice applications of the Hölder inequality.

The author's search in Chapter 8 for non-negative solutions of (1) on $(0,\infty)$ satisfying (in one case),

$$\lim_{t\to 0^+} p(t)y'(t) = 0, \qquad y(t) \to 0, \text{ as } t \to \infty, \tag{11}$$

is motivated by the classical problem of finding positive solutions of Poisson's equation in $\mathbf{R}^n$ reduced to finding radial solutions to

$$u'' + \frac{n-1}{r}u' + h(u) = 0, \quad 0 < r < \infty,$$

and satisfying $u'(0) = 0$, and $u(r) \to 0$, as $r \to \infty$. It is assumed here that $f : [0,\infty) \times \mathbf{R}^2 \to \mathbf{R}$ is continuous, $q \in C(0,\infty)$, $p \in C[0,\infty) \cap C^{(1)}(0,\infty)$, both $p > 0$ and $q > 0$ on $(0,\infty)$, and

$$\int_0^{a_0} p(x)q(x)dx < \infty \text{ and } \int_0^{a_0} \frac{1}{p(s)} \int_0^s p(x)q(x)dx\, ds < \infty,$$

for each $a_0 > 0$. In addition, also assumed are $f(t,0,0) \leq 0$, for all $t \geq 0$, there exists $r_0 > 0$ such that $f(t,r_0,0) \geq 0$, for all $t \geq 0$, and a Nagumo type condition. Consideration of a corresponding one-parameter family in the spirit of previous arguments leads to, for each $N \in \mathbf{N}$, a non-negative solution $y_N(t)$ of (1) on $(0,N)$ satisfying

$$\lim_{t\to 0^+} p(t)y'(t) = 0, \qquad y(N) = 0,$$

and such that $0 \leq y_N(t) \leq r_0$ on $[0,N]$. To discuss the boundary condition in (11) at infinity (that is, to pass to the limit with the sequence $\{y_N(t)\}$), cases are considered. The first case deals with $p(t) = t^\gamma$, $\gamma > 1$. The arguments, while tedious, are provided in entirety, and an illustrative example is given with the singular BVP,

$$y'' + \frac{\gamma}{t}y' = (y^n - e^{-t})t^{-1/2}, \quad 0 < t < \infty,$$

satisfying (11), where $n \geq 0$, $\gamma > 1$.



Chapter 9 introduces singular BVP's in which the singularity occurs in the dependent variable as is the case in pseudoplastic fluids and some boundary layer theory. Positive solutions are sought for

$$y'' + f(t,y) = 0, \quad 0 < t < 1, \tag{12}$$

satisfying

$$-\alpha y(0) + \beta y'(0) = 0, \quad \alpha^2 + \beta^2 > 0, \qquad ay(1) + by'(1) = 0, a^2 + b^2 > 0, \tag{13}$$

with $\alpha, \beta, a, b \geq 0$ and $\alpha + a > 0$, where $f : [0,1] \times (0,\infty) \to (0,\infty)$ is continuous. For the case when

$$0 < f(t,y) \leq Ay + h(y) + By^{-\gamma}$$

on $(0,1) \times (0,\infty)$, $A$, $B$, $\gamma \geq 0$ and $h \geq 0$ on $(0,\infty)$, and $y^\gamma h(y) \leq Cy^{\gamma+\tau} + D$, for $y > 0$ and some $C$, $D \geq 0$, $0 \leq \tau \leq 1$, *a priori* bounds are obtained for solutions of an associated one-parameter family of equations and satisfying

$$-\alpha y(0) + \beta y'(0) = \frac{\alpha}{n}, \qquad ay(1) + by'(1) = \frac{a}{n}, \quad n \in \mathbf{N},$$

so that the Non-linear Alternative can be applied to obtain a positive solution $y_n(t)$, for each $n \in \mathbf{N}$. It is then assumed that there exist $M > 0$ and $\psi \in C[0,1]$ which is positive on $(0,1)$ such that $f(t,x) \geq \psi(t)$ on $(0,1) \times (0,M]$. An application of Arzela-Ascoli yields a solution of (13) and (14). This method also extends to obtain solutions of (13) satisfying many of the other boundary conditions of the book.

The last chapter, Chapter 10, is devoted to the existence of positive solutions of

$$\frac{1}{p}(py')' = \varphi^2 q f(t,y,y'), \quad 0 < t < 1, \tag{14}$$

$$y(0) = a > 0, \quad y(1) = b \geq 0, \tag{15}$$

where $p$ may have singularities at $t = 0$ and/or 1, and $f$ may be singular at $y = 0$, in that

$$f : [0,1] \times (0,\infty) \times (-\infty,\infty) \to (0,\infty)$$

is continuous, $\lim_{y\to 0^+} f(x,y,v) = +\infty$ uniformly on compact subsets of $[0,1] \times (-\infty,\infty)$, and

$$f(t,y,v) \le [g(y) + h(y)]k(v),$$

where $g$ is continuous, positive and non-increasing, and $h \ge 0$, $k > 0$ are both continuous on $[0,\infty)$. The most interesting case for (14), (15) is when $b = 0$. This is addressed by considering the appropriate one-parameter homotopy family for (14) satisfying

$$y(0) = a > 0, \quad y(1) = \frac{a}{n}, \quad n \in \mathbb{N}, \tag{16}$$

making sufficient assumptions so that *a priori* bounds are obtained for this associated family of BVP's, again obtaining positive solutions $y_n(t)$, and then passing to the limit, with Arzela-Ascoli providing a positive solution of (14), (15), with $b = 0$.

This is a very readable and attractive book, containing much basic information and with a contemporary outlook on singular BVP's for second order ODE's. The references give an adequate sample of the relevant literature on this topic.

# A PROBLEM OF BOURBAKI ON FIELD THEORY

Rod Gow

The following problem appeared in one of Bourbaki's early chapters on algebra, [1, p.146]. Let $K$ be a commutative field of characteristic different from 2 and let $f$ be a mapping of $K$ into itself such that

$$f(x+y) = f(x) + f(y)$$

for all $x$ and $y$ in $K$ and

$$f(x)f(x^{-1}) = 1$$

for all non-zero $x$. Show that $f$ is an isomorphism of $K$ onto a subfield of $K$ (or alternatively, a monomorphism of $K$). In other words, we must show that

$$f(xy) = f(x)f(y)$$

for all $x$ and $y$.

In fact, Bourbaki's result is not strictly true as it stands. For it follows from the relation $f(x)f(x^{-1}) = 1$ that $f(1)^2 = 1$ and thus $f(1) = \pm 1$. Now if $f(1) = -1$, $f$ is not a monomorphism, but it can be proved that $-f$ defined by $(-f)(x) = -f(x)$ is a monomorphism. We will assume throughout this discussion that $f(1) = 1$. We note that Bourbaki's exercise was still being presented in the incorrect form in later editions such as [2, p.175].

A hint is given in Bourbaki's exercise: show that $f(x^2) = f(x)^2$ for all $x$ (there is a misprint of this in [1]). It took us some time to prove the equality above and, to allow people to try to prove this for themselves, if they so wish, we will not present our