Outline Solutions of the Problems for the 35th IMO

1. Without loss of generality $a_1 < a_2 < \dots < a_m$. Suppose $a_i + a_{m+1-i} \leq n$, for some i with $1 \leq i \leq m$. Then $a_j + a_{m+1-i} \leq n$, for $j = 1, 2, \dots, i$. But then the i distinct integers $a_j + a_{m+1-i}$, $j = 1, 2, \dots, i$ must lie in the set $\{m, m-1, \dots, m-i+2\}$, which contains only $i - 1$ elements. Thus $a_i + a_{m+1-i} \geq n + 1$, for $i = 1, 2, \dots, m$. Add these inequalities to obtain the result.

2. Use coordinates. Without loss of generality, let $M = (0,0)$, $B = (-1,0)$ $C = (1,0)$. Let $A = (0,a)$ and $Q = (t,0)$. The rest of the solution is straightforward.

3. (*a*) Let A_k be the set of integers in $\{1, 2, \dots, k\}$ whose base 2 representation contains exactly three 1's and let $g(k)$ be the number of elements in A_k. Then f and g are nondecreasing functions and $f(k) = g(2k) - g(k)$. Then

$$f(k+1) - f(k) = g(2k+2) - g(2k) - (g(k+1) - g(k)).$$

Now either both $2k+2 \in A_{2k+2}$ and $k+1 \in A_{k+1}$ or neither is true. Thus $f(k+1) - f(k) = 0$ or 1, depending on whether $2k+1 \in A_{2k+1}$ or not. Thus $f(k)$ does not skip any positive integer values. Since

$$g(2^n) = \binom{n}{3} = g(2^n - 1),$$

we get, after some calculation, $f(2^n) = \binom{n}{2}$. Thus f is not bounded above and hence assumes every non-negative integer value.

(*b*) Suppose $f(k) = m$ has a unique solution. Then

$$f(k+1) - f(k) = 1 = f(k) - f(k-1).$$

The former holds if and only if $2k+1 \in A_{2k+2}$, i.e. there are exactly two 1's in the base 2 digits of k. The same holds for $k-1$.

This is possible if and only if $k-1$ has exactly two 1's in its base 2 representation, where the last digit is 1 and the second last digit is 0, i.e. $k = 2^n + 2$ for some integer $n \geq 2$. A calculation gives

$$f(2^n + 2) = \binom{n}{2} + 1.$$

Thus the set of positive integers m for which $f(k) = m$ has a unique solution is $\{\binom{n}{2} + 1 : n \geq 2\}$.

4. We note that

$$\frac{n^3+1}{mn-1} + 1 = \frac{n(n^2+m)}{mn-1}$$

and that

$$\frac{m(n^2+m)}{mn-1} - n = \frac{m^2+n}{mn-1}.$$

Thus $mn - 1$ divides $n^3 + 1$ if and only if it divides $m^2 + n$ and this holds if and only if $mn - 1$ divides $m^3 + 1$.

If $m = n$ it is easy to see that $m = 2$.

If $m > n$, then $\frac{n^2+m}{mn-1} = k$, an integer, implies that $n^2 + k = m(kn-1) > kn^2 - n$ and thus $(k-1)n^2 - n - k < 0$. This implies that $n < \frac{k}{k-1}$, if $k > 1$.

If $k = 1$, then $n^2 + m = mn - 1$. Thus $m = n + 1 + \frac{2}{n-1}$. The fact that $n - 1$ divides 2 proves that $n = 2$ or 3. If $n = 2$, then $m = 5$ and if $n = 3$ then $m = 5$.

If $k > 1$, then $n < \frac{k}{k-1} \leq 2$ implies that $n = 1$. Then $m = 2$ or 3.

Thus, if $\frac{n^3+1}{mn-1}$ is an integer, (m,n) is one of the pairs:

$$(1,2), (1,3), (2,1), (3,1), (2,5), (3,5), (5,2), (5,3), (2,2).$$

It is clear that $\frac{n^3+1}{mn-1}$ is an integer if (m,n) is one of these nine pairs.

5. It is clear that $\frac{f(x)}{x}$ can take the value 1 at most once in each of the intervals (-1,0) and $(0, \infty)$. Let $f(a) = a$, then property (*i*) implies that $f(2a+a^2)=2a+a^2$. If $-1<a<0$, then $-1<2a+a^2<0$ and thus $a=2a+a^2$. This gives the contradiction $a=0$ or -1. Similarly, the assumption that $a>0$ leads to a contradiction. Thus $f(a)=a$ implies $a=0$. Using this fact and letting $y=x$ in (i) proves

$$x+f(x)+xf(x)=0,$$

for all x in S. Thus

$$f(x)=\frac{-x}{1+x}$$

for all x in S. It is clear that this function satisfies (i) and (ii) and is the only function with these two properties.

6. First solution. Let A be the set of all positive integers of the form $q_1q_2\dots q_{q_1}$, where $q_1<q_2<\dots<q_{q_1}$ are primes. For any infinite set $\{p_1,p_2,p_3,\dots\}$ of primes $p_1<p_2<p_3<\dots$, we can satisfy the requirements of the problem, by taking

$$m=p_1p_2\dots p_{p_1} \text{ and } n=p_2p_3\dots p_{p_1+1}.$$

Second solution. Let $\Pi=\{p_1,p_2,p_3,\dots\}$ denote the set of all primes. Let

$$A_i=\{q_1q_2\dots q_i : q_1,q_2,\dots,q_i\in\Pi \text{ and } p_i \not| q_1q_2\dots q_i\}$$

and let $A=A_1\cup A_2\cup A_3\cup\dots$. Let S be any infinite subset of Π and let p_k be in S. Choose distinct primes $q_1,q_2,\dots,q_k$ in $S-\{p_k\}$. Then $m=q_1q_2\dots q_{k-1}q_k$ is in A, whereas $n=q_1q_2\dots q_{k-1}p_k$ is not in A.

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