RECENT RESULTS ON THE ORDER STRUCTURE OF COMPACT OPERATORS

Y. A. Abramovich and A. W. Wickstead

Abstract: There have been only a few really positive results concerning the order structure of spaces of compact operators on Banach lattices, although many related open questions have been posed over the years. Recent results by the authors show why this is so—those few positive results describe virtually all that is true!

1 Introduction.

We will consider linear operators between real Banach lattices. A Banach lattice is a Banach space, E, which is also a vector lattice with the two structures related by the implication $|x| \leq |y| \Rightarrow \|x\| \leq \|y\|$ for all $x, y \in E$. If E and F are Banach lattices then we will be considering various subspaces of the spaces $\mathcal{L}(E,F)$ of bounded linear operators, and $\mathcal{K}(E,F)$ of compact linear operators.

If E and F are Banach lattices then an operator $T : E \to F$ is termed *positive* if $x \geq 0 \Rightarrow Tx \geq 0$. The linear span of the positive operators is the space of *regular* operators, denoted by $\mathcal{L}^r(E,F)$. Usually $\mathcal{L}^r(E,F)$ is a proper subspace of $\mathcal{L}(E,F)$. When we order $\mathcal{L}^r(E,F)$ by defining $S \geq T \Leftrightarrow S - T$ is positive, the space $\mathcal{L}^r(E,F)$ is certainly made into a partially ordered vector space, but is not, in general, a lattice. The most important case in which it is a lattice is when F is Dedekind complete, i.e. when every non-empty set with an upper bound must have a least upper bound (the corresponding assertion for lower bounds follows automatically). In this case $\mathcal{L}^r(E,F)$ is itself a Dedekind complete vector lattice. The basic results of the theories of Banach

lattices and of positive operators may be found in [8], [29] or [37] as well as in several other texts.

One reason for studying compact operators is that there seemed, at least for some years, a good chance that the order structure of, say, the compact regular operators might be rather better than that of regular operators. There are of course other reasons for their study. The work of Krengel that we describe in the next section had its origins in ergodic theory, whilst the Dodds-Fremlin theorem has applications in theoretical physics (see [11]). A reasonable hope might have been that the compact operators between two Banach lattices formed a lattice under the operator order. This turns out not to be true but slightly lower expectations would still seem to be reasonable. These hopes turned out to be forlorn in general. There are some partial positive results, for the statements of which we need to give a few definitions.

In this survey, we have concentrated solely on the order structure of spaces of compact operators. There are many other topics in the theory of compact positive operators which lie off the main direction chosen for this survey and on which substantial progress has been made in recent years. Topics that we could have mentioned include results in [10], [13], [14], [17], [18], [21], [35], [36] and [39] on factorizing compact positive operators; the vast literature on the spectral theory of compact positive operators; a special and important part of the latter stemming from the Andô-Krieger theorem and culminating in de Pagter's proof, in [33], that an irreducible compact positive operator has strictly positive spectral radius and in further refinements obtained in [3]; and many other areas.

2. Some Banach lattice terminology.

There are two special classes of Banach lattices that have been studied almost as long as Banach lattices themselves. An *AM-space* is a Banach lattice in which $\|x \vee y\| = \|x\| \vee \|y\|$ whenever $x, y \geq 0$. It was shown in [23] and in [25] that each AM-space is isometrically order isomorphic to a closed sublattice of some space $C(K)$ where K is a compact Hausdorff space. An *AL-space* is a

Banach lattice in which $\|x+y\| = \|x\| + \|y\|$ whenever $x, y \geq 0$. Again it was shown in [22] that each AL-space is isometrically order isomorphic to an $L^1(\mu)$-space for some measure μ.

The normed dual, E', of a Banach lattice E may be naturally ordered by defining $f \geq g \Leftrightarrow f(x) \geq g(x)$ for all $0 \leq x \in E$. Under this order E' is also a Banach lattice (and is even Dedekind complete). The concepts of AM- and AL-spaces are mutually dual, i.e. the dual of an AM-space is an AL-space whilst the dual of an AL-space is an AM-space.

A subspace J of a Banach lattice E is an *ideal* whenever $x \in E, y \in J$ and $|x| \leq |y|$ imply that $x \in J$. A *band* is an ideal with the extra property that if a subset of J has a supremum in E then that supremum must actually lie in J.

A Banach lattice has an *order continuous norm* if every downward directed family with infimum equal to zero must converge in norm to zero. An equivalent condition is that the Banach lattice be an ideal in its bidual. Banach lattices with an order continuous norm are Dedekind complete. If $1 \leq p < \infty$ then each space $L^p(\mu)$ has an order continuous norm. Spaces $C(K)$ have an order continuous norm only if K is a finite set.

A Banach lattice is a *KB-space* (=Kantorovich-Banach space) if it has an order continuous norm, and every norm-bounded upward directed set has a supremum. Various equivalences of this are known. One is that the Banach lattice is weakly sequentially complete, another is that it be a band in its bidual. All the spaces $L^p(\mu)$, for $1 \leq p < \infty$ are KB-spaces. The space of all null-sequences, c_0, with the supremum norm and the pointwise ordering, is an example of a Banach lattice which has an order continuous norm but which is not a KB-space.

An *atom* in a Banach lattice is a non-zero positive element e such that if $0 \leq x \leq e$ then x is a multiple of e. A Banach lattice is *atomic* if for every $0 < x \in E$ there is an atom e with $e \leq x$. $L^p(\mu)$ is atomic if and only if μ is a discrete measure, whilst $C(K)$ is atomic precisely when K has a dense subset of isolated points. Below, at some point we will meet the notion of atomic Banach lattices with an order continuous norm. Archetypal examples are ℓ_p, for $1 \leq p < \infty$ and c_0.

3. Krengel's results.

There are several possible questions that one might ask about the order structure of a subspace $\mathcal{I}$ of the space of bounded operators. We certainly want to know whether or not $\mathcal{I}$ is positively generated and whether or not it is a lattice. Furthermore if $\mathcal{I}$ is a lattice, then we want to know whether or not the lattice operations in $\mathcal{I}$ are also the corresponding lattice operations in the space of all regular operators. In general we cannot talk about $\mathcal{I}$ being a sublattice of the space of all regular operators as the latter space need not be a lattice (unless, for example, the range is Dedekind complete). There is no suitable terminology in the literature describing this situation; so it seems reasonable to extend the definition of sublattices as follows. If $\mathcal{J}$ is a partially ordered vector space and $\mathcal{I}$ a subspace of $\mathcal{J}$ then we say that $\mathcal{I}$ is a (*generalized*) *sublattice* of $\mathcal{J}$ if $\mathcal{I}$ is a lattice and for each $x, y \in \mathcal{I}$ the supremum of x and y calculated in $\mathcal{I}$ is also their supremum in $\mathcal{J}$. In the case when $\mathcal{J}$ is a vector lattice this is the usual definition of a sublattice. We will often omit the adjective "generalized" unless it is necessary to emphasize that the ambient space is not a lattice.

Using the properties of the compact subsets of $C(K)$-spaces, Krengel, in [26], established the first positive result in this area by proving the following.

Theorem 3.1. [Krengel] *If E is an arbitrary Banach lattice and F an arbitrary AM-space then $\mathcal{K}(E,F)$ is a generalized sublattice of $\mathcal{L}^r(E,F)$.*

In particular this means that every compact operator taking values in an AM-space does have a modulus (in $\mathcal{L}^r(E,F)$) and this modulus is again compact. A simple duality argument establishes a similar result if E is an AL-space and F is a KB-space. Later we shall see that this result can be improved somewhat.

In [27], Krengel gave two important examples, showing that these results are certainly not true in general. Much work in this area since then has been devoted to trying to rescue some remnant of his earlier positive results for spaces different from AM-spaces.

Example 3.2. [Krengel] *There is a Dedekind complete Banach lattice E and a compact operator T on E such that $|T|$ exists in $\mathcal{L}^r(E)$, but is not compact.*

The crucial feature of the corresponding construction is as follows. Consider a $2^n \times 2^n$ matrix with orthogonal rows and with all entries being ± 1. If this matrix is regarded as an operator S_n on 2^n-dimensional Hilbert space, E_n, then $\|S_n\| = 2^{n/2}$ whilst $\||S_n|\| = 2^n$. The required example is produced by taking a suitable weighted sum of the operators S_n acting on $\left(\sum E_n\right)_{c_0}$, the c_0-sum of the spaces E_n, whose elements are sequences (x_n) with $x_n \in E_n$ and $\|x_n\| \to 0$. All the examples subsequently produced in this field are based on modifications of various degrees of complexity of this construction.

This example shows already that even on Dedekind complete Banach lattices, the compact operators do not form a sublattice of the lattice of all regular operators. Although the operator T in Example 3.2 is a regular operator, it is easy to verify that T is not the difference of two compact positive operators, so that the space of compact operators on E is not positively generated. In fact a modification of the previous example shows that the compact operators are not even a subspace of $\mathcal{L}^r(E)$.

Example 3.3. [Krengel] *There is a Dedekind complete Banach lattice E and a compact operator T on E such that $|T|$ does not exist in $\mathcal{L}^r(E)$.*

Compact regular operators taking values in a Dedekind complete Banach lattice must have a modulus as all regular operators then do, so that the operator in Example 3.3 is not regular. Krengel's examples left open the possibility that for compact regular operators the condition of Dedekind completeness could be dropped. However, using Krengel's basic finite-dimensional building blocks, we constructed in [6] an example to show that this also was false.

Example 3.4. [Abramovich & Wickstead] *There is a Banach lattice E and a compact regular operator T on E such that $|T|$ does not exist in $\mathcal{L}^r(E)$.*

In the example that we constructed in [6] the operator T is *not* the difference of two positive compact operators. This led us to the believe that the *correct* space to study was the linear span of the positive compact operators rather than the space of all compact operators. We denote the space of differences of positive compact operators from E into F by $\mathcal{K}^r(E,F)$. Thus, $\mathcal{K}^r(E,F)$ is a subspace of $\mathcal{K}(E,F) \cap \mathcal{L}^r(E,F)$. By either looking at Krengel's proof of Example 3.2 or using its statement together with the Dodds-Fremlin theorem (Theorem 4.1 below), one can see that even when F is assumed to be Dedekind complete $\mathcal{K}^r(E,F) \neq \mathcal{K}(E,F) \cap \mathcal{L}^r(E,F)$. Although this space $\mathcal{K}^r$ must surely be better behaved than, say, the space of compact regular operators $\mathcal{K}(E,F) \cap \mathcal{L}^r(E,F)$, it does not behave all that well. In [7] we, again using Krengel's basic building blocks, produced two more examples.

Example 3.5. [Abramovich & Wickstead] *There is a Banach lattice E and $T \in \mathcal{K}^r(E)$ which does not have a modulus either in $\mathcal{K}^r(E)$ or in $\mathcal{L}^r(E)$.*

If E were Dedekind complete then operators in $\mathcal{K}^r(E)$ must certainly have a modulus in $\mathcal{L}^r(E)$, however...

Example 3.6. [Abramovich & Wickstead] *There is a Dedekind complete Banach lattice E and $T \in \mathcal{K}^r(E)$ which does not have a modulus in $\mathcal{K}^r(E)$.*

In particular the modulus of T, computed in $\mathcal{L}^r(E)$ is not compact.

There seems to be little left to conjecture as being true in great generality. How much further can we extend Krengel's positive results? His proof of Theorem 3.1 actually establishes that in that case $\mathcal{K}(E,F)$ is a Banach lattice under the operator order and the usual operator norm (there is another norm used in the study of regular operators, but we have no need of it in this survey). There are few cases where this holds that Krengel did not already deal with. The following theorem is due to Krengel [26], Cartwright and Lotz [12], and Schwarz [38, Theorem 8.1]. A simple proof of Schwarz's contribution will appear in [42].

Theorem 3.7. *If E and F are Banach lattices then $\mathcal{K}(E,F)$ is a Banach lattice under the operator order and norm if, and only if, either E is an AL-space or F is an AM-space.*

Certainly if either E is isomorphic to an AL-space or F is isomorphic to an AM-space then $\mathcal{K}(E,F)$ is isomorphic to a Banach lattice. However, there is no isomorphic version of Theorem 3.7. As pointed out in [2], it follows from an example in [1] and Theorem in [19] that the next result is true.

Example 3.8. *There are Dedekind complete Banach lattices E and F such that E is not isomorphic to an AL-space and F is not isomorphic to an AM-space but $\mathcal{K}(E,F)$ coincides with $\mathcal{K}^r(E,F)$ and is isomorphic to a Banach lattice.*

4. The Dodds-Fremlin theorem and its consequences.

Apart from Krengel's examples, little positive had been known about the order structure of spaces of compact operators until Dodds and Fremlin published their now celebrated theorem in [15]. In retrospect it is clear that there are many antecedents of this result in the literature, including [28], [31], [24, Theorem 5.10], [34] and [37, Theorem 10.2], but at the time the result came to most people as a complete surprise.

Theorem 4.1. [Dodds & Fremlin] *If E and F are Banach lattices such that both E' and F have order continuous norms, $T : E \to F$ is a positive compact linear operator and $S : E \to F$ is a linear operator such that $0 \le S \le T$ then S is compact.*

We refer to the conclusion of this theorem as the *compact domination property*. The Dodds-Fremlin condition (i.e. the continuity of norms in E' and F) is not the only one that guarantees the compact domination property. The other two, found in [40], are very strong conditions and the proofs of the compact domination property in these latter cases is rather simple. The conditions are that E' (resp. F) be atomic *and* have an order continuous norm. Some important, but easily deduced, consequences of the compact domination property are the following:

(α) If F is Dedekind complete (which is automatic if the Dodds-Fremlin condition holds) then $\mathcal{K}^r(E,F)$ is an (order) ideal in $\mathcal{L}^r(E,F)$ and therefore $\mathcal{K}^r(E,F)$ is a Dedekind complete vector lattice. In particular if $S,T : E \to F$ are two positive compact operators then $S \vee T$, which automatically exists in $\mathcal{L}^r(E,F)$, is compact and thus belongs to $\mathcal{K}^r(E,F)$.

(β) If F is only assumed to be Dedekind σ-complete then $\mathcal{K}^r(E,F)$ is a Dedekind σ-complete vector lattice. The only reason we cannot say that $\mathcal{K}^r(E,F)$ is an ideal in $\mathcal{L}^r(E,F)$ is that the latter space need not be a lattice.

Even in as apparently nice a context as that of operators into an AM-space, no analogue of the first conclusion in (α) is true, i.e. $\mathcal{K}^r(E,F)$ may easily fail to be Dedekind complete. To demonstrate this let E be an AL-space and X be a compact Hausdorff space. It has been known since [32] that $C(X)$ is Dedekind complete if and only if X is *Stonean*, i.e. the closure of every open subset of X is again open. There is an isometric order isomorphism between $\mathcal{K}(E,C(X))$ and $C(X,E')$. Since E' is a Dedekind complete AM-space, it can be identified with a space $C(Y)$ for some compact Hausdorff Stonean space Y. We may now identify $C(X,E')$ with $C(X,C(Y))$ and hence with $C(X \times Y)$ for both the norm and order structure. It follows from a well-known, but unpublished, result of W. Rudin (see [16] for a short proof using order theoretic notions) that $X \times Y$ is Stonean only when one factor is finite and the other Stonean. Thus as long as both E and $C(X)$ are infinite dimensional the space $K(E,C(X))$ cannot be Dedekind complete (or even Dedekind σ-complete).

There is an obvious interest in extending these known conditions which guarantee the compact domination property. At first sight it looks plausible that there is a whole spectrum of conditions that will do, with the two extremes being when either E' or F is atomic with an order continuous norm, and the Dodds-Fremlin theorem simply identifying an easily described case somewhere in the middle of the range. However, as recently shown in [41], that is not the case, and this makes the Dodds-Fremlin result all the more remarkable.

Theorem 4.2. [Wickstead] *The pair of Banach lattices E and F has the compact domination property if and only if one of the following three non-exclusive conditions holds:*

(a) Both E' and F have an order continuous norm.
(b) E' is atomic and has an order continuous norm.
(c) F is atomic and has an order continuous norm.

It is similarly surprising that the compact domination property is not just a simple sufficient condition for proving the two consequences (α) and (β) mentioned above. The following two results are proved in [42].

Theorem 4.3. [Wickstead] *If E and F are Banach lattices then $\mathcal{K}^r(E,F)$ is a Dedekind complete vector lattice if and only if the pair (E,F) has the compact domination property and F is Dedekind complete.*

Theorem 4.4. [Wickstead] *If E and F are Banach lattices then $\mathcal{K}^r(E,F)$ is a Dedekind σ-complete vector lattice if and only if the pair (E,F) has the compact domination property and F is Dedekind σ-complete.*

Notice now that the second conclusion in (α), that the supremum of two positive compact operators exists and is compact, is not an equivalence of the compact domination property. For example, by Theorem 3.1, this is also the case whenever F is an AM-space. Moreover, in all previously known cases the supremum of two positive compact operators was always compact whenever it existed, in particular whenever F was Dedekind complete. This led C. D. Aliprantis and O. Burkinshaw to ask for a counterexample to or a proof of this phenomenon. The question was posed by them at a *Riesz Spaces and Operator Theory* meeting at Oberwolfach in 1982, and reiterated in [20, Problem 6]. Unfortunately, the answer is negative; namely the examples in [6] show the following:

Example 4.5. [Abramovich & Wickstead] *There is a Dedekind complete Banach lattice E and compact positive operators $S,T : E \to E$ such that $S \vee T$ is not compact.*

Example 4.6. [Abramovich & Wickstead] *There are Banach lattices E and F and compact positive operators $S,T : E \to F$ such that $S \vee T$ does not exist in $\mathcal{K}^r(E,F)$.*

Before leaving this section we should mention, although the results are not directly related to the study of the order structure of spaces of compact operators, the extensions of the Dodds-Fremlin theorem proved by C. D. Aliprantis and O. Burkinshaw, in their remarkable work [9]. They managed to find some "hidden" compactness in positive operators dominated by a compact positive operator by proving the following theorem.

Theorem 4.7. [Aliprantis & Burkinshaw] *Let $0 \le S \le T$ be two positive operators on a Banach lattice E and assume that T is compact. Then operator S^3 is compact. If either E or E' has order continuous norm, then S^2 is compact.*

This theorem has many applications, of which the most interesting are in connection with the spectral properties of positive operators [3], [33] and with the invariant subspace problem for positive operators [4], [5]. For some applications in connection with positive semigroups we refer to [30] and references therein.

5. What is left to prove?

Although many conjectures have now been disposed of, there do remain some open questions in this area. We have characterized the cases in which $\mathcal{K}^r(E,F)$ is either a Dedekind complete or Dedekind σ-complete vector lattice. There is still no answer known to:

Question 5.1. *For what pairs of Banach lattices E and F is $\mathcal{K}^r(E,F)$ a vector lattice?*

The answer will certainly *not* be that the pair satisfy the compact domination property, since the conclusion also holds whenever F is an AM-space or E is an AL-space. On the other hand Example 4.6 shows that $\mathcal{K}^r(E,F)$ may fail to be a vector lattice. In all the cases that we do have a lattice, $\mathcal{K}^r(E,F)$ is a generalized sublattice of $\mathcal{L}^r(E,F)$. Possibly the following question might be rather more tractable.

Question 5.2. *For what pairs of Banach lattices E and F is $\mathcal{K}^r(E,F)$ a generalized sublattice of $\mathcal{L}^r(E,F)$?*

However both of these questions seem rather difficult at present. Perhaps the best that we might hope for will be an answer to:

Question 5.3. *If $\mathcal{K}^r(E,F)$ is a vector lattice, must it be a generalized sublattice of $\mathcal{L}^r(E,F)$?*

If it is so difficult for spaces of compact operators to have a lattice structure, is there some useful weaker order theoretic structure that we can look for? The *Riesz separation property* states that if $x_1, x_2 \leq z_1, z_2$ then there is y with $x_1, x_2 \leq y \leq z_1, z_2$. This condition is (slightly) weaker than that of being a lattice but has some important consequences. For example it, together with a fairly natural condition relating the norm and order, are equivalent to the dual of an ordered normed space being a Banach lattice under the dual ordering. In [43] the second author showed that it is possible to find Banach lattices E and F such that $\mathcal{L}^r(E,F)$ has the Riesz separation property, but is not a lattice. It is also possible to choose E and F such that $\mathcal{L}^r(E,F)$ does not have the Riesz separation property. The proofs used in [43] do not answer the corresponding questions for $\mathcal{K}^r(E,F)$, so these questions remain open. It is probably too much to expect an answer to:

Question 5.4. *For what pairs of Banach lattices E and F does $\mathcal{K}^r(E,F)$ have the Riesz separation property?*

But we would certainly hope for answers to the next two questions. In particular we feel that the answer to Question 5.6 is almost certainly positive.

Question 5.5. *Are there Banach lattices E and F such that $\mathcal{K}^r(E,F)$ has the Riesz separation property, but is not a lattice?*

Question 5.6. *Are there Banach lattices E and F such that $\mathcal{K}^r(E,F)$ does not have the Riesz separation property?*

References

[1] Y. A. Abramovich, *On the space of operators between Banach lattices*, Zapiski LOMI **73** (1977), 188–192.

[2] Y. A. Abramovich, *When each regular operator is continuous* in Functional Analysis, Optimization and Mathematical Economics, Oxford University Press: New York, 1990, 133–140.

[3] Y. A. Abramovich, C. D. Aliprantis and O. Burkinshaw, *On the spectral radius of positive operators*, Math. Z. **211** (1992), 593–607.

[4] Y. A. Abramovich, C. D. Aliprantis and O. Burkinshaw, *Invariant subspaces of positive operators*, J. Funct. Analysis (forthcoming).

[5] Y. A. Abramovich, C. D. Aliprantis and O. Burkinshaw, *Invariant subspaces for positive operators acting on a Banach space with basis*, Proc. Amer. Math. Soc. (forthcoming).

[6] Y. A. Abramovich and A. W. Wickstead, *A compact regular operator without modulus*, Proc. Amer. Math. Soc. **116** (1992), 721–726.

[7] Y. A. Abramovich and A. W. Wickstead, *Solutions of several problems in the theory of compact positive operators* (submitted).

[8] C. D. Aliprantis and O. Burkinshaw, Positive Operators. Academic Press: New York & London, 1985.

[9] C. D. Aliprantis and O. Burkinshaw, *Positive compact operators on Banach lattices*, Math. Z. **174** (1980), 289–298.

[10] C. D. Aliprantis and O. Burkinshaw, *Factoring compact and weakly compact operators through reflexive Banach lattices*, Trans. Amer. Math. Soc. **183** (1984), 369–381.

[11] J. Avron, I. Herbst and B. Simon, *Schrödinger operators with magnetic fields, I. General interactions*, Duke Math. J. **45** (1978), 847–883.

[12] D. I. Cartwright and H. P. Lotz, *Some characterizations of AM- and AL-spaces*, Math. Z. **142** (1975), 97–103.

[13] Z. L. Chen, *The factorization of a class of compact operators*, Dongbei Shuxue **6** (1990), 303–309.

[14] W. J. Davis, T. Figiel, W. B. Johnson and A. Pelczynski, *Factoring weakly compact operators*, J. Funct. Anal. **17** (1974), 311–327.

[15] P. G. Dodds and D. H. Fremlin, *Compact operators in Banach lattices*, Israel J. Math. **34** (1979), 287–320.

[16] Z. Ercan and A. W. Wickstead, *Banach lattices of continuous Banach lattice-valued functions* (submitted).

[17] T. Figiel, *Factorization of compact operators and applications to the approximation property*, Studia Math. **45** (1973), 191–210.

[18] N. Ghoussoub and W. B. Johnson, *Factoring operators through Banach lattices not containing $C(0,1)$*, Math. Z. **194** (1987), 153–171.

[19] G. Godefroy, *Sur les opérateurs compacts réguliers*, Bull. Soc. Belg. A **30** (1978), 35–37.

[20] C. B. Huijsmans and W. A. J. Luxemburg (eds.), Positive Operators and Semigroups on Banach Lattices. Kluwer: Dordrecht, 1992.

[21] W. B. Johnson, *Factoring compact operators*, Israel J. Math. **9** (1971), 337–345.

[22] S. Kakutani, *Concrete representations of abstract (L)-spaces and the mean ergodic theorem*, Ann. of Math. **42** (1941), 523–537.

[23] S. Kakutani, *Concrete representation of abstract (M)-spaces*, Ann. of Math. **42** (1941), 994–1024.

[24] M. A. Krasnoselskii, P. P. Zabreiko, E. I. Pustylnik and P. E. Sobolevskii, Integral Operators in Spaces of Summable Functions. Noordhoff Intnl. Publ.: Leiden, 1976.

[25] M. Krein and S. Krein, *On an inner characterisation of the set of all continuous functions defined on a bicompact Hausdorff space*, C. R. Acad. Sci. URSS **27** (1940), 427–430.

[26] U. Krengel, *Über den Absolutbetrag stetiger linearer Operatoren und seine Anwendung auf ergodische Zerlegungen*, Math. Scand. **13** (1963), 151–187.

[27] U. Krengel, *Remark on the modulus of compact operators*, Bull. Amer. Math. Soc. **72** (1966), 132–133.

[28] W. A. J. Luxemburg & A. C. Zaanen, *Compactness of integral operators in Banach function spaces*, Math. Ann. **149** (1963), 422–447.

[29] P. Meyer-Nieberg, Banach Lattices. Springer-Verlag: Berlin, Heidelberg, New York, 1991.

[30] M. Mokhtar-Kharroubi, *Compactness properties for positive semigroups on Banach lattices and applications*, Houston J. Math. **17** (1991), 25–38.

[31] R. J. Nagel and U. Schlotterbeck, *Kompaktheit von integraloperatoren auf Banachverbänden*, Math. Ann. **202** (1973), 301–306.

[32] H. Nakano, *Über das System aller stetigen Funktionen auf einem topologischen Raum*, Proc. Imp. Acad. Tokyo **17** (1940–41), 308–310.

[33] B. de Pagter, *Irreducible compact operators*, Math. Z. **192** (1986), 149–153.

[34] L. D. Pitt, *A compactness condition for linear operators on function spaces*, J. Operator Theory **1** (1979), 49–54.

[35] I. M. Popovici and D. T. Vuza, *Factoring compact operators and approximable operators*, Z. Anal. Anwendungen **9** (1990), 221–233.

[36] I. M. Popovici and D. T. Vuza, *Factorization of convolution operators*, An. Univ. Craiova Mat. Fiz.-Chim. **16** (1988), 1–8.

[37] H. H. Schaefer, Banach Lattices and Positive Operators. Springer: Berlin & New York, 1974.

[38] H. U. Schwarz, Banach Lattices and Operators. Teubner Texte: Leipzig, 1984.

[39] M. Talagrand, *Some weakly compact operators between Banach lattices do not factor through reflexive Banach lattices*, Proc. Amer. Math. Soc. **96** (1986), 95–102.

[40] A. W. Wickstead, *Extremal structure of cones of operators*, Quart. J. Math. Oxford Ser. 2 **32** (1981), 239–253.

[41] A. W. Wickstead, *Converses for the Dodds-Fremlin and Kalton-Saab theorems* (submitted).

[42] A. W. Wickstead, *Dedekind completeness of some lattices of compact operators* (submitted).

[43] A. W. Wickstead, *Spaces of operators with the Riesz Separation Property* (submitted).

Y. A. Abramovich
Department of Mathematics
IUPUI, 402 Blackford Street
Indianapolis, IN 46202
USA

A. W. Wickstead
Department of Pure Mathematics
The Queen's University of Belfast
Belfast BT7 1NN