If $Z(G)$ is finite we are finished, so assume that $Z(G)$ is infinite. Then

$$Z(G) \simeq T \times C_\infty \times \cdots \times C_\infty$$

is the direct product of a finite group T and finitely many infinite cyclic groups. Now $Z(G)$ is invariant under all epimorphisms of G onto G, but clearly $\alpha : x \to x^{n+1}$ is not onto on any of the infinite cyclic factors. This contradiction establishes the result.

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THE POISSON KERNEL AS AN EXTREMAL FUNCTION

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It has long been known that for certain classical inequalities involving positive harmonic functions on the open unit ball B of $\mathbf{R}^N$ the Poisson kernel of B (with some fixed pole on ∂B) is extremal (that is, a function for which equality holds in the inequalities). In recent years several new inequalities for positive harmonic functions on B have been discovered; again the Poisson kernel and functions related to it appear in extremal roles. This article, which is based on part of a talk given at the Society's Meeting at Waterford in September 1992, surveys some such inequalities, both old and new.

1. Harmonic functions and the Poisson kernel

1.1. *Harmonic functions*

A real-valued function h is *harmonic* on a non-empty open subset Ω of the Euclidean space $\mathbf{R}^N$, where $N \geq 2$, if h is smooth (that is, $h \in C^2(\Omega)$) and satisfies Laplace's equation (that is, $\Delta h \equiv 0$ on Ω, where $\Delta = \partial^2/\partial x_1^2 + \ldots + \partial^2/\partial x_N^2$). Harmonic functions are also characterized by Gauss' mean value property: h is harmonic on Ω if and only if h is continuous on Ω and, for each closed ball $\beta \subset \Omega$, the value of h at the centre of β is equal to its average value over the boundary $\partial\beta$ of β (see, e.g., Hayman and Kennedy [13, §1.5.5]).

For ease of reference, we list the classes of harmonic functions that we shall consider:

$$\begin{aligned} H_N &= \{h : h \text{ is harmonic on } B\}, \\ H_N^+ &= \{h \in H_N : h > 0 \text{ on } B, h(0) = 1\}, \\ HH_{m,N} &= \{h : h \text{ is a homogeneous harmonic polynomial} \\ &\qquad \text{of degree } m \text{ on } \mathbf{R}^N\}, \\ H_{m,N} &= \{h : h \text{ is a harmonic polynomial of degree at} \\ &\qquad \text{most } m \text{ on } \mathbf{R}^N\}, \\ H_{m,N}^+ &= \{h \in H_{m,N} : h > 0 \text{ on } B, h(0) = 1\}. \end{aligned}$$

Clearly the spaces $H_N, HH_{m,N}$ and $H_{m,N}$ are real vector spaces ($0 \in HH_{m,N}$ by convention). The normalization $h(0) = 1$ in the definitions of H_N^+ and $H_{m,N}^+$ is convenient and involves no real loss of generality.

1.2 *The Poisson kernel*

The Poisson integral and hence implicitly the Poisson kernel for B were introduced in the 1820's (Poisson [16], [17]) for $N = 2,3$ in a construction aimed at solving (what later became known as) the Dirichlet problem for B. The *Poisson kernel* K of B is defined on $B \times \partial B$ by

$$K(x,y) = (1 - \|x\|^2)\|x-y\|^{-N}, \tag{1.2.1}$$

where $\|\cdot\|$ is the Euclidean norm on $\mathbf{R}^N$. A calculation (see, e.g., [13, p.32]) shows that if $y \in \partial B$ is fixed, then $K(\cdot, y) \in H_N^+$. If μ is a finite signed measure on ∂B, then the Poisson integral P_μ of μ is defined on B by

$$P_\mu(x) = \int_{\partial B} K(x,y)d\mu(y). \tag{1.2.2}$$

Passing the operator Δ under the integral sign in (1.2.2), we deduce from the harmonicity of the functions $K(\cdot, y)$ that $P_\mu \in H_N$; if, further, μ is a probability measure on ∂B (that is $\mu \geq 0$ and $\mu(\partial B) = 1$), then $P_\mu \in H_N^+$. The importance of the Poisson integral lies partly in the converse result which, following Doob [7, 1.II.4], we call the *Riesz-Herglotz representation theorem:*

if $h \in H_N^+$, then $h = P_\mu$ for some probability measure μ on ∂B.

(A proof can also be found, e.g., in Helms [14, Theorem 2.13].)

We explain in passing the connection between Poisson integrals and the Dirichlet problem, mentioned above. If $\mu = f\sigma$, where $f : \partial B \to \mathbf{R}$ is continuous on ∂B and σ is surface measure on ∂B normalized so that $\sigma(\partial B) = 1$, then $P_\mu \in H_N$ and $P_\mu(x) \to f(y)$ as $x \to y$ for all $y \in \partial B$; that is to say, P_μ solves the classical Dirichlet problem for B with boundary function f.

Our aim here is to illustrate the extremal role of the Poisson kernel in relation to inequalities involving three classes of functions.

2. Inequalities for positive harmonic functions

We mention three inequalities (two classical, one recent) for functions of class H_N^+.

2.1 *Harnack's inequalities*

It is easy to see that

$$(1-\|x\|)(1+\|x\|)^{1-N} \leq K(x,y) \leq (1+\|x\|)(1-\|x\|)^{1-N} \tag{2.1.1}$$

for all $x \in B$ and all $y \in \partial B$. If $h \in H_N^+$, then, by the Riesz-Herglotz theorem, $h = P_\mu$ for some probability measure μ on ∂B. Integrating each member of (2.1.1) with respect to $d\mu(y)$, we obtain the Harnack inequalities [12]:

For $h \in H_N^+, x \in B$,

$$(1-\|x\|)(1+\|x\|)^{1-N} \leq h(x) \leq (1+\|x\|)(1-\|x\|)^{1-N}. \tag{2.1.2}$$

For (2.1.2) the Poisson kernel is extremal: more precisely, examining cases of equality in (2.1.1), we find that if $x \in B\backslash\{0\}$, the left-hand (respectively, right-hand) inequality in (2.1.2) is strict unless $h = K(\cdot, y)$ for some $y \in \partial B$ and $x = -\alpha y$ (respectively, $x = \alpha y$) for some $\alpha \in (0,1)$.

2.2 *A corollary of Harnack's inequalities*

From (2.1.2) it follows easily that if $h \in H_N^+$, then

$$\|x\|^{-1}|h(x) - h(0)| \le N + O(\|x\|) \qquad \text{as } \|x\| \to 0,$$

whence

$$\|\nabla h(0)\| \le N \qquad (h \in H_N^+), \tag{2.2.1}$$

where ∇ is the gradient operator: $\nabla = (\partial/\partial x_1, \ldots, \partial/\partial x_N)$. In particular,

$$|(\partial h/\partial x_1)(0)| \le N \qquad (h \in H_N^+). \tag{2.2.2}$$

Calculations show that equality holds in (2.2.1) if $h \in K(\cdot, y)$ for some $y \in \partial B$ and in (2.2.2) if $h = K(\cdot, y)$ with $y = (\pm 1, 0, \ldots, 0)$; with a little more trouble one can show that these are the only cases of equality.

2.3 *A generalization of* (2.2.2.)

Goldstein and Kuran [10] generalized (2.2.1) and (2.2.2). As a sample of their work, we state a generalization of (2.2.2) in the case $N = 3$: for $m = 1, 2, \ldots$

$$-m!(2m+1) \le (\partial^m h/\partial x_1^m)(0) \le m!(2m+1) \quad (h \in H_3^+); \tag{2.3.1}$$

moreover, equality holds on the right hand side if and only if $h = K(\cdot, (1,0,0))$. (The question of sharpness in the left-hand inequality is another story; see [3].) A proof of (2.3.1) is out of the question here, but the idea is to write h as a Poisson integral, pass the operator $\partial^m/\partial x_1^m$ under the integral sign in (1.2.2), and then (the hard part) estimate $(\partial^m/\partial x_1^m)K(x,y)$ at $x = 0$. In §4.1 we indicate the significance of the factor $2m+1$ in (2.3.1).

3. Inequalities for trigonometric polynomials

3.1 *Trigonometric polynomials and harmonic polynomials*

To study trigonometric polynomials is essentially to study plane harmonic polynomials, as we now explain. We identify $\mathbf{R}^2$ with $\mathbf{C}$ in the usual way. For a positive integer j, the functions $\mathrm{Re}(z^j)$ and $\mathrm{Im}(z^j)$ span $HH_{j,2}$. Thus a typical element h of $H_{m,2}$ has the form

$$h(re^{i\theta}) = a_0 + \sum_{j=1}^{m} r^j (a_j \cos j\theta + b_j \sin j\theta), \tag{3.1.1}$$

where the coefficients are real. Let T_m denote the space of real-valued trigonometric polynomials (defined on the unit circle) of degree at most m. A typical element f of T_m can be written as

$$f(e^{i\theta}) = a_0 + \sum_{j=1}^{m} (a_j \cos j\theta + b_j \sin j\theta). \tag{3.1.2}$$

The isomorphism $\Phi : T_m \to H_{m,2}$ mapping the function in (3.1.2) to that in (3.1.1) clearly maps each $f \in T_m$ to the solution of the Dirichlet problem in the unit disc D with boundary function f. Let

$$T_m^+ = \{f \in T_m : f \ge 0 \text{ on } \partial D, \int_0^{2\pi} f(e^{i\theta})\, d\theta = 2\pi\}.$$

Note that the elements of $H_{m,2}^+$ and T_m^+ are normalized so that $a_0 = 1$ in the representations (3.1.1) and (3.1.2) respectively. It follows from the well-known minimum principle for harmonic functions that $\Phi(T_m^+) = H_{m,2}^+$. Thus results for T_m^+ can be interpreted for $H_{m,2}^+$.

3.2 *An inequality of Fejér*

Fejér [8] proved that

$$\sup_{\partial D} f \le m + 1 \qquad (f \in T_m^+). \tag{3.2.1}$$

Here is a quick proof. Write $f \in T_m^+$ in the form (3.1.2). Since $f \geq 0$ on ∂D and $a_0 = 1$, for all θ we have

$$\begin{aligned} f(e^{i\theta}) &\leq \sum_{k=0}^{m} f(e^{i(\theta+2k\pi/(m+1))}) \qquad (3.2.2) \\ &= m+1+ \\ &\sum_{j=1}^{m}\sum_{k=0}^{m}\left\{a_j \cos\left(j\theta + \frac{2jk\pi}{m+1}\right) + b_j \sin\left(j\theta + \frac{2jk\pi}{m+1}\right)\right\} \\ &= m+1; \end{aligned}$$

the last step uses the equation

$$\sum_{k=0}^{m} e^{2ijk\pi/(m+1)} = 0 \quad (j = 1, \ldots, m).$$

It is easy to see that in working with $f \in T_m^+$, there is no real loss of generality in supposing that $\sup_{\partial D} f = f(1)$. It was also shown by Fejér that there is exactly one $f \in T_m^+$ such that $\sup_{\partial D} f = f(1) = m+1$, and this function is

$$f_m(e^{i\theta}) = 1 + 2(m+1)^{-1}\sum_{j=1}^{m}(m+1-j)\cos j\theta.$$

(A proof can be based on the observations that equality holds in (3.2.2) with $\theta = 0$ if and only if $f(e^{2ik\pi/(m+1)}) = 0$ for all $k = 1, \ldots, m$ and that f_m has this property, since

$$f_m(e^{i\theta}) = (m+1)^{-1}\sin^2((m+1)\theta/2)/\sin^2(\theta/2) \qquad (0 < \theta < 2\pi).)$$

Interpreted for harmonic polynomials, these results say that

$$\sup_{\partial D} h \leq m+1 \qquad (h \in H_{m,2}^+) \qquad (3.2.3)$$

and

$$h_m(re^{i\theta}) = 1 + 2(m+1)^{-1}\sum_{j=1}^{m}(m+1-j)r^j \cos j\theta \qquad (3.2.4)$$

is the only element of $H_{m,2}^+$ for which $\sup_{\partial D} h = h(1) = m+1$. To see how the extremal functions h_m are related to the Poisson kernel of D, note that writing $N = 2, x = re^{i\theta}, y = 1$ in (1.2.1) gives

$$K(re^{i\theta}, 1) = (1-r^2)(1+r^2-2r\cos\theta)^{-1} = 1 + \sum_{j=1}^{\infty} 2r^j \cos j\theta \qquad (3.2.5)$$

and that $h_m(re^{i\theta})$ is the mth Cesàro (that is, $(C,1)$) mean of this series (including the term 1); in particular, $h_m \to K(\cdot, 1)$ locally uniformly on D as $m \to \infty$.

3.3 *An inequality of Szegö*

If $f \in T_m^+$ is given by (3.1.2) (so that $a_0 = 1$), how big can the individual coefficients a_j, b_j be? For simplicity consider only a_j. A crude estimate is easy:

$$0 \leq \pi^{-1}\int_0^{2\pi}(1 \pm \cos j\theta)f(e^{i\theta})d\theta = 2 \pm a_j,$$

so that $|a_j| \leq 2$. Szegö's sharp result [18, p.625] is

$$|a_j| \leq 2\cos(\pi/(2+[m/j])), \qquad (3.3.1)$$

where $[\cdot]$ is the integer part function. Much later Kuran and I (unpublished) rediscovered the harmonic polynomial version : if $h \in H_{m,2}^+$ and h is given by (3.1.1), then a_j satisfies (3.3.1); note that $j!a_j = (\partial^j h/\partial x_1^j)(0)$. We were also able to say something about extremal functions. For example, with $j = 1$:

$$|(\partial h/\partial x_1)(0)| \leq 2\cos(\pi/(2+m)) \qquad (h \in H_{m,2}^+) \qquad (3.3.2)$$

(compare (2.2.2) with $N = 2$), and there is exactly one function $h_m \in H^+_{m,2}$ for which equality holds in (3.3.2) with the modulus sign suppressed; further $h_m \to K(\cdot, 1)$ locally uniformly on D as $m \to \infty$.

3.4 *A recent result and an open question*
A question of Holland [1, Problem 4.26] essentially asks for the value of

$$\Lambda_m = \sup_{f\in T^+_m} \frac{1}{2\pi}\int_0^{2\pi} (f(e^{i\theta}))^2\, d\theta = \sup_{h\in H^+_{m,2}} \frac{1}{2\pi}\int_0^{2\pi} (h(e^{i\theta}))^2\, d\theta.$$

No general formula for Λ_m has been found, but it is known that the suprema are attained (Goldstein and McDonald [11])and that $\lim \Lambda_m/m$ exists and equals $0.68698\ldots$ (Garsia et al. [9]; see also Brown et al. [6]). It would be interesting to know whether the extremal functions are again related to the Poisson kernel of D: if $h_m \in H^+_{m,2}$ is such that $\sup_{\partial D} h_m = 1$ and

$$\Lambda_m = \frac{1}{2\pi}\int_0^{2\pi} (h_m(e^{i\theta}))^2\, d\theta,$$

do we have $h_m \to K(\cdot, 1)$ on D as $m \to \infty$? The conjecture that (Λ_m/m) is decreasing also seems to be open.

4. Inequalities for harmonic polynomials

4.1 *General remarks about harmonic polynomials*
In their guise as questions about harmonic polynomials, the problems discussed in §3 can be posed in all dimensions. They are generally more difficult in higher dimensions, since complex variable techniques are not readily available and representations of harmonic polynomials on $\mathbf{R}^N$ are not as simple as (3.1.1) when $N \geq 3$.

We need to quote some well-known facts; Brelot and Choquet [5] give an excellent account of most of these. Let u denote the unit vector $(1, 0, \ldots, 0)$ in $\mathbf{R}^N$. There is exactly one element $I_{m,N}$ of $HH_{m,N}$ such that $I_{m,N}$ is x_1-axial (that is, $I_{m,N}$ depends only on x_1 and $||x||$) and $I_{m,N}(u) = 1$. Writing $\cos\theta = x_1/||x||$ when $x \in \mathbf{R}^N\backslash\{0\}$, we have, for example,

$$I_{m,2}(x) = ||x||^m \cos m\theta, \quad I_{m,3} = ||x||^m P_m(\cos\theta),$$

where P_m is the Legendre polynomial of degree m. Also, we have $|I_{m,N}| \leq 1$ on ∂B and

$$\int_{\partial B} I^2_{m,N}\, d\sigma = 1/d_{m,N}, \tag{4.1.1}$$

where

$$d_{m,N} = \dim HH_{m,N} = \frac{2m+N-2}{m+N-2}\binom{m+N-2}{N-2}, \tag{4.1.2}$$

so that for $m \geq 1$

$$d_{m,2} = 2, \quad d_{m,3} = 2m+1, \quad d_{m,4} = (m+1)^2.$$

The N-dimensional generalization of (3.2.5) is

$$K(x,u) = 1 + \sum_{j=1}^{\infty} d_{j,N} I_{j,N}(x) \qquad (x \in B) \tag{4.1.3}$$

(see Müller [15, Lemma 17]). It is the appearance of the coefficients $d_{j,N}(= 2j+1$ when $N = 3)$ in (4.1.3) that ultimately accounts for the factor $2m+1$ in (2.3.1).

4.2 *An axialization technique*
We explain a device which greatly simplifies some extremal problems for harmonic polynomials. If $f : \mathbf{R}^N \to \mathbf{R}$ is continuous, then we define its x_1-*axialization* $f^* : \mathbf{R}^N \to \mathbf{R}$ by writing $f^*(x) = f(x)$ if x lies on the x_1-axis and otherwise defining $f^*(x)$ to be the average value of f on the set $\{y : y_1 = x_1, ||y|| = ||x||\}$. It turns out that if h is harmonic, then so also is h^*; moreover if $h \in HH_{m,N}$, then $h^* = h(u)I_{m,N}$ (see [2, Lemma] for details). This observation

allows us to reduce the proofs of several inequalities for harmonic polynomials to consideration of x_1-axial harmonic polynomials.

4.3 *A generalization of (3.2.3)*
Kuran and I [4] obtained explicit, best possible constants $C_{m,N}$ such that

$$\sup_{\partial B} h \leq C_{m,N} \qquad (h \in H^+_{m,N}).$$

We have $C_{m,2} = m + 1$ (see (3.2.3)) and

$$C_{m,N} \sim 2^{2-N}((N-1)!)^{-1} m^{N-1}$$

as $m \to +\infty$ with N fixed. Our proof depends on the technique of §4.2 and known quadrature formulae for ultraspherical (Legendre when $N = 3$) polynomials. As in the plane case (§3.2), there is a unique extremal element h_m of $H^+_{m,N}$ which satisfies the equation $\sup_{\partial B} h = h_m(u) = C_{m,N}$. We have no simple formula, corresponding to (3.2.4), for h_m when $N \geq 3$, but it remains true in all dimensions that $h_m \to K(\cdot, u)$ locally uniformly on B as $m \to \infty$.

4.4 *A generalization of (3.3.2)*
Szegö [18, p.626] obtained the following analogue of (3.3.2) for the case $N = 3$:

$$|(\partial h/\partial x_1)(0)| \leq 3\tau_m \qquad (h \in H^+_{m,3}), \tag{4.4.1}$$

where τ_m is the greatest zero of $P_{(m+2)/2}$ or $P_{(m+1)/2} + P_{(m+3)/2}$ according as m is even or odd. Note that $\tau_m \in [0,1)$ and $\tau_m \to 1$ as $m \to \infty$, and compare (4.4.1) with the case $N = 3$ of (2.2.2). Techniques like those mentioned in §4.3 (axialization and quadrature formulae) can be used to generalize (4.4.1) to all dimensions. Again the extremal polynomials for the N-dimensional generalization of (4.4.1) are related to the Poisson kernel.

4.5 *Two norms on $H_{m,N}$*
For polynomials P, Q on $\mathbf{R}^N$, let

$$< P, Q > = \int_{\partial B} PQ \, d\sigma,$$
$$||P||_2 = \sqrt{< P, P >},$$
$$||P||_\infty = \sup_{\partial B} |P|.$$

Note that $< \cdot, \cdot >$ is not an inner product and $|| \cdot ||_2, || \cdot ||_\infty$ are not norms on the space of all such polynomials, for a polynomial may vanish on ∂B but be not identically zero. However $< \cdot, \cdot >$ is an inner product and $|| \cdot ||_2, || \cdot ||_\infty$ are norms on $H_{m,N}$. Clearly $||P||_2 \leq ||P||_\infty$ for all polynomials P. An inequality in the opposite direction for polynomials of degree at most m is

$$||P||_\infty \leq \sqrt{d_{m,N+1}} ||P||_2, \tag{4.5.1}$$

and this is sharp. We briefly explain how the constant $\sqrt{d_{m,N+1}}$ comes to appear in (4.5.1). First note that $||P||_2$ and $||P||_\infty$ involve only the values of P on ∂B, and there is an element of $H_{m,N}$ which agrees with P on ∂B (see [5]). Hence we may suppose that $P \in H_{m,N}$. By a rotation of axes, we may further suppose that $||P||_\infty = |P(u)|$, and an argument based on the observations in §4.2 allows us to suppose that P is x_1-axial. Then P has the representation

$$P = a_0 I_{0,N} + a_1 I_{1,N} + \ldots + a_m I_{m,N},$$

so that

$$||P||_\infty = |P(u)| = |a_0 + \ldots + a_m|. \tag{4.5.2}$$

Further, by (4.1.1) and the orthogonality relation

$$< I_{j,N}, I_{k,N} > = 0 \qquad (0 \leq j < k),$$

we obtain

$$||P||_2 = \sqrt{\sum_{j=0}^{m} a_j^2 / d_{j,N}}. \tag{4.5.3}$$

In view of (4.5.2) and (4.5.3), the Cauchy-Schwarz inequality gives

$$||P||_\infty \leq \sqrt{d_{0,N} + \ldots + d_{m,N}}||P||_2.$$

A calculation using (4.2.2) shows that $d_{0,N}+\ldots+d_{m,N} = d_{m,N+1}$, and (4.5.1) now follows.

Checking for cases of equality at each stage of the argument, we find without much difficulty that if $P \in H_{m,N}$ and equality holds in (4.5.1), then $P = \alpha K_m$ for some real α, where K_m is the mth partial sum of the series expansion (4.1.3) of $K(\cdot, u)$, that is

$$K_m = 1 + \sum_{j=1}^{m} d_{j,N} I_{j,N}.$$

It is hoped that details of (4.5.1) and some related inequalities will appear elsewhere.

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