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cumann matamaitice na héireann

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The Bulletin seeks articles of mathematical interest written in an expository style. All areas of mathematics are welcome, pure and applied, old and new. The Bulletin is typeset using TEX. Authors are invited to submit their articles in the form of TEX input files if possible, in order to ensure speedier processing.

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2. The current subscription fees are given below.

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8. Any application for membership must be presented to the Committee of the I.M.S. before it can be accepted. This Committee meets twice each year.

9. Please send the completed application form with one year's subscription fee to

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MAURICE KENNEDY
Obituary

It was with great regret that the Irish mathematical community learned of the death, following a long, serious illness, of Professor Maurice Kennedy on 15 January 1994. He had retired, on grounds of ill-health, from University College Dublin in 1983. At the time of his retirement he held the offices of Registrar and Associate Professor of Mathematics. He is survived by his brother and his sister.

Maurice Kennedy was born in Dublin in 1924, the son of a distinguished figure of the early days of the new Irish State, Dr Henry Kennedy (an active member of the cooperative movement). Maurice received his secondary education at Belvedere College and he won an Entrance Scholarship to University College Dublin in 1942. He spent his first year studying engineering, but switched to Mathematical Science in his second year, graduating with a first class honours B.Sc. in 1945. He gained his M.Sc. in 1946 and was appointed an assistant in the departments of mathematics and mathematical physics in 1947. In 1951 he went to the California Institute of Technology on a Smith-Mundt Scholarship. Although he used frequently bemoan the inadequacy of his mathematical background as a preparation for graduate study, he completed his Ph.D. degree under Samuel Karlin in three years with a thesis entitled *Ergodic theorems for a certain class of Markov processes*. These results were later published in the Pacific Journal of Mathematics [**3**]. Maurice returned to Ireland in 1954 and took up an appointment as an assistant lecturer in mathematics in UCD. Apart from a sabbatical year in Stanford university in 1958–59 he spent the rest of his academic life in University College Dublin, becoming College Lecturer in 1959, Associate Professor in 1966 and Registrar in 1974.

When Maurice was a graduate student at Caltech he did work related to the famous Bateman Manuscript Project. Harry Bateman had been a professor of mathematics there and, at the time of his death in 1946, had been working on a large project to give an up-to-date account of the properties of the special functions of mathematical physics. The results of his work were contained on a card index file occupying some dozens of shoe boxes. Arthur Erdélyi had been appointed to a professorship at Caltech with the specific task of completing the Bateman project. The results of all this work appeared as *Higher Transcendental Functions* (three volumes) and *Tables of Integral Transforms* (two volumes) and were published by McGraw-Hill, beginning in 1953. The work that Maurice did in this connection was done as part of his duties as a research assistant and was published in [1, 2]. Inspired by his year in Stanford, where he again met Samuel Karlin, Maurice published a second paper on stochastic processes [4]

Professor Kennedy was an outstanding teacher and was particularly inspiring to weaker students. His presentations were meticulous in every detail. He pioneered the teaching of measure theory, functional analysis and stochastic processes at UCD. At a different level he was involved in giving courses to school teachers to help them implement changes in the school curriculum [5]. In the early 70s he organized a memorable series of seminars based on the book *Geometry of Quantum Theory* (vol. 1) by V. S. Varadarajan (Van Nostrand, 1970). His idea was to get a topic which would contain mathematics that would appeal to the widest possible audience. As well as lecturing himself, he always summed up and commented on the lectures of the other speakers. In his later years at UCD he took a very great interest in general topology and foundations of analysis and produced some results in these areas which were never published.

Maurice Kennedy became very active in all aspects of College life. He had very deeply held views on the nature of the university, he had very high standards and he argued his views at all levels of the College, from Arts Faculty to Governing Body. He was deeply suspicious of the academic value of a number of the newer disciplines. Since 1972 he was a member of the Senate of the National

University of Ireland and served on its Standing Committee and Board of Studies. From 1978 to 1980 he acted as executive secretary of the Committee of Heads of Irish Universities and represented the Committee at meetings in Brussels, Strasbourg and Helsinki. Almost all his waking hours were devoted to the work of UCD. However, he did have an abiding interest in Irish politics, as befitted someone who came from a family that was heavily involved in political life in the early days of the State. His one recreation was grand opera and he never missed the opportunity to attend a performance if he was visiting a major European city.

What distinguished Maurice Kennedy's participation in academic life was his deep understanding of the nature of the university as an academic institution. Even those who disagreed with him respected the dedication and integrity with which he defended and promoted academic ideals and values. No matter whom he was dealing with he never compromised his values for any short-term gain.

Publications

1. (With A. Erdélyi and J. L. McGregor) Parabolic cylinder functions of large order, J. Rational Mech. Anal.3, 459-485 (1954).

2. (With A. Erdélyi and J. L. McGregor) Asymptotic forms of Coulomb wave functions I, Tech. Report 4, Department of Mathematics, California Institute of Technology, Pasadena, 1955, 29pp.

3. A convergence theorem for a certain class of Markov processes, Pacific J. Math. 7(1957), 1107-1124.

4. A stochastic process associated with the ultraspherical polynomials, Proc. Roy. Irish Acad., Sect. A, 61(1961), 89-100.

5. (With R. Ingram, S. O'Brien and J. R. Timoney) Mathematics for Teachers, Department of Mathematics, University College Dublin, 1963.

Minutes of the Meeting of the Irish Mathematical Society

Ordinary Meeting

21st December 1993

The Irish Mathematical Society held an ordinary meeting at 12.15 pm on Tuesday 21st December 1993 at the DIAS, 10 Burlington Road. 15 members were present. The president, B. Goldsmith, was in the chair.

1. The **minutes** of the meeting of 8th April 1993 were approved and signed.

2. There were no **matters arising**.

3. The following **correspondence** was received:
(i) A letter from the Mathematics Fund For Bosnia-Hercegovina asking for financial contributions from societies and individuals. It was decided that the IMS should not contribute.
(ii) A circular from I. Halperin about the Campaign For Human Rights - An End To Apartheid (South Africa).
(iii) Two letters regarding the Bulletin and two letters regarding the new draft constitution (see below).

4. **Bulletin**
The Committee has received and accepted a letter of resignation from the editor, J. Ward. The President thanked him for his services. R. Gow has been appointed as the new editor.

The current issue of the Bulletin has been distributed to most individual members of the Society, but 100 more copies need to be printed to cover institutional members and journal exchanges. These will be printed in Bolton Street.

The Committee has decided not to use the Eolas printers in future. P. Mellon had obtained a quote of £690 for printing 400 copies of the Bulletin by O'Brien Printing Ltd, but that company is now defunct. G. Lessells has obtained a quote of £1 per copy

for a similar sized batch from the University of Limerick printers. G. Lessells and E. Gath will be responsible for printing and distributing of the next issue of the Bulletin.

It was remarked that the Bulletin is the most important function of the Society, and that once it is running smoothly the Society should be able to increase its annual membership fees.

Members who are currently refereeing articles for the Bulletin are urged to send their reports to R. Gow as soon as possible.

5. **IMS Constitution**
A new draft constitution and draft rules have been circulated to members with the last issue of the Bulletin. D. Tipple proposed that the meeting adopt these. This was seconded by M. Ó Searcóid and unanimously accepted by the meeting. D. Tipple then proposed certain amendments to the new constitution and rules. The amendments were seconded by M. Ó Searcóid and unanimously accepted by the meeting.

6. **European Mathematical Society**
B. Goldsmith has been invited to join the EMS subcommittee on mathematics for developing countries. S. Dineen announced that there will be a meeting of the EMS next August in Zürich.

7. **Treasurer's business**
The treasurer reported that the Society has around 240 members and that the annual fee is £10 (or $10 for reciprocity members). Eolas support has diminished over the years, and will possibly be nonexistent next year. The income for next year may therefore be as low as £2400. The Society currently has around £1800 in the bank. In view of this unhealthy financial situation it will be necessary to reduce the Society's support for conferences in the immediate future. He reiterated the suggestion that, once the Bulletin is running smoothly, membership fees may need to be raised.

The President thanked the treasurer for his hard work over the last four years.

8. **Elections**

The following were elected, unopposed, to the Committee (Re-election to the Committee is denoted by $*$):

Committee member	Proposed	Seconded
P. Mellon* (Secretary)	R. Timoney	M. Ó Searcóid
M. Vandyck (Treasurer)	D. Hurley	J. Pulé
E. Gath*	J. Pulé	G. Lessells
R. Gow	M. Ó Searcóid	D. Tipple
C. Nash	M. Ó Searcóid	P. Mellon
R. Timoney*	B. Goldsmith	S. Dineen

The following have one more year of office: B. Goldsmith (President), D. Hurley (Vice-President), G. Lessells, B. McCann, M. Ó Searcóid, J. Pulé.

The Committee is to co-opt one member from UCG.

The following have left the Committee: G. Ellis, D. Tipple

The President thanked the out-going Secretary for his services over the last four years.

9. **September meeting**

S. Dineen and S. Gardiner are organizing the 1994 annual meeting at UCD for 5th and 6th September. It will be followed by a three day conference on polynomials and holomorphic functions. Accommodation will be available on the UCD campus at £13.50 per night.

The 1995 annual meeting will be held at the University of Limerick.

10. There was no **other business**.

The meeting closed at 1.00 pm.

Graham Ellis
University College
Galway

A VIEWPOINT ON MINIMALITY IN TOPOLOGY

P. T. Matthews and T. B. M. McMaster

Introduction.

Given a family $\mathcal{F}$ of topological spaces whose point-sets all have the same cardinality, and a particular space X in $\mathcal{F}$, what should we mean by saying that X is *minimal* in $\mathcal{F}$, and what use can be made of such a concept?

Well, it depends both on the nature of the family and, critically, on the ordering relation between the spaces which belong to it. If for example we take $\mathcal{F}$ as a collection of spaces all having the same underlying point-set S, and order them by refinement of topology (writing $(S,\tau_1) \leq (S,\tau_2)$ if and only if every τ_1-open set is τ_2-open) then we are looking at part of the lattice of topologies on S. Here the *interpretation* of minimality is entirely unambiguous and straightforward: (S,τ) is minimal in $\mathcal{F}$ if, whenever $(S,\tau') \leq (S,\tau)$ and $(S,\tau') \in \mathcal{F}$, then $\tau = \tau'$. The *techniques* required, however, to access minimal objects in this context can on occasions be extremely complex and subtle (see, for instance, Larson [3], Johnston and McCartan [5,6], McCluskey [10], McCluskey and McCartan [11,12,13]) and the resulting insights correspondingly deep: indeed, as has been persuasively argued, "In seeking to identify those [topologies on S] which minimally satisfy an invariant property, we are, in a very real sense, examining the topological essence of the invariant" [11].

Nevertheless there are aspects of general topology for which this approach to minimality is not appropriate. It is often the correct practice (especially when the discussion is in any sense categorical) to co-identify spaces which are homeomorphic to one another, in effect working with the homeomorphism classes in preference to the multitude of examples within each class. This device is not readily compatible with 'refinement of topology' since it is easy to exhibit on any infinite set two homeomorphic topologies one of which is strictly finer than the other, and in any case the 'same underlying point-set' phrase is rendered meaningless by focussing on classes. A different ordering relation is therefore needed here, and the one most readily available is that of 'embeddability as a subspace', the binary relation *sub* defined by X sub Y iff X is homeomorphic to a subspace of Y. This has several desirable features, such as respecting homeomorphic equivalence, relating nicely to hereditary classes of spaces, and being reflexive and transitive. But it is not antisymmetric (the open and closed intervals (0,1) and [0,1] are by no means the same space, yet (0,1) sub [0,1] and [0,1] sub (0,1) are both true) and herein lies the difficulty: how can we assign a meaning to minimality of elements in a set which is not *partially* ordered but only *quasi*-ordered? And, of course, why should we bother to do so?

This note considers two suggestions for answering the first question. One is effectively that adopted in Ginsburg and Sands' paper [2] and in the unresolved 'Toronto problem' which is associated with it. We use the other to establish a proposition, previously unnoticed so far as we have been able to determine, about that best-known of all topological spaces, the real line; it will then be seen to play a key role in characterizing the circumstances in which Bankston's 'Anti-' operation [1] exhibits a certain behaviour. Hopefully these findings will be perceived as a partial answer to the second question above!

We thank the referee for helpful and perceptive criticisms of this article.

Strong and weak quasi-minimality.

Take an infinite cardinal α, $\mathcal{T}(\alpha)$ to denote the family of all topological spaces on α points, $\mathcal{F}$ a subfamily of $\mathcal{T}(\alpha)$, X a member of $\mathcal{F}$ and sub as described in the Introduction. Let us agree to call X *strongly quasi-minimal in* $\mathcal{F}$ if

$$Y \text{ sub } X, Y \in \mathcal{F} \text{ imply } Y \text{ homeomorphic to } X,$$

and *weakly quasi-minimal in* $\mathcal{F}$ if

$$Y \text{ sub } X, Y \in \mathcal{F} \text{ imply } X \text{ sub } Y.$$

The abbreviations sqm and wqm will be employed, and the following remarks are immediate:

Proposition.

(i) *sqm in $\mathcal{F}$ implies wqm in $\mathcal{F}$ (for any $\mathcal{F}$).*

(ii) *The converse is not always valid (consider the two-space counterexample* $\mathcal{F} = \{(0,1), [0,1]\}$).

(iii) *In any $\mathcal{F}$ which is partially ordered by sub (after identification of homeomorphic spaces) the sqm, wqm and minimal elements coincide.*

Of particular interest for our applications is the case $\mathcal{F} = \mathcal{T}(\alpha)$, so we shall compactify our notation further and write 'X is sqm' (or wqm) rather than 'X is sqm in $\mathcal{T}(\alpha)$' (or wqm in $\mathcal{T}(\alpha)$). So an sqm space is homeomorphic to each of its equicardinal subspaces, a wqm space is embeddable into each of its equicardinal subspaces. What do these spaces look like?

Well, in the case $\alpha = \aleph_0$ Ginsburg and Sands give a complete and remarkably tidy answer [2]. They observe that on the set of positive integers the discrete, trivial, cofinite, initial-segment and final-segment topologies give sqm spaces, and they demonstrate that every infinite space contains a copy of one or more of these five (let us call them GS spaces).

Theorem (Ginsburg and Sands).

(i) In $\mathcal{T}(\aleph_0)$ the sqm and the wqm spaces are precisely the five GS spaces.

(ii) $\mathcal{T}(\aleph_0)$ is "supported" by its wqm members, in the sense that for each X in $\mathcal{T}(\aleph_0)$ there is some wqm Y in $\mathcal{T}(\aleph_0)$ such that Y sub X.

Corollary. *In the class $T_2 \cap \mathcal{T}(\aleph_0)$ of denumerable Hausdorff spaces, only the discrete space is sqm (or wqm) and this space on its own supports $T_2 \cap \mathcal{T}(\aleph_0)$.*

At present it is far from clear what happens to these results when $\aleph_0$ is replaced by a larger cardinal. Certainly not every uncountable Hausdorff space contains a discrete equicardinal subspace, so for part (ii) of the theorem to be generalizable to $\aleph_1$ or beyond we should require uncountable non-discrete Hausdorff sqm spaces; the Toronto problem [14, p.15] asks whether such objects exist, and it has yet to be answered. Our contribution to the debate is to observe that if they do exist, then there are "not enough of them to support their colleagues" in the above sense, even if we relax sqm to wqm. More precisely, we show (subject to set-theoretic assumptions) that for some uncountable cardinals α,

(a) the wqm spaces in $\mathcal{T}(\alpha)$ do not support $\mathcal{T}(\alpha)$,

(b) the wqm spaces in $T_2 \cap \mathcal{T}(\alpha)$ do not support $T_2 \cap \mathcal{T}(\alpha)$,

(c) any subfamily of $\mathcal{T}(\alpha)$ which does support the entire family must have fairly large cardinality.

The proof, embodied largely in the following three lemmas, consists of a transfinite-induction construction combined with a variant of a standard argument relating the weight of a space to the number of its autohomeomorphisms.

Lemma A. *Let α be a regular cardinal, X a set of cardinality α, and $\{S_\beta : \beta < \alpha\}$ a family of α subsets of X each having α elements. Then there is an α-element subset T of X which contains none of the sets S_β.*

Proof: Suppose for convenience that X *is* α. We define a strictly increasing transfinite sequence $(x_\beta, \beta < \alpha)$ in such a way that $x_\beta \in S_\beta$ and $x_{\beta'} > (x_\beta)'$ for each β, the "dash" indicating successor in α. Initialize by choosing

x_0= the least element of S_0.

Now assuming (for typical non-zero $\beta < \alpha$) that the x_γ for $\gamma < \beta$ have been chosen in accordance with the desired criteria, we note that the set $\{x_\gamma : \gamma < \beta\}$ has smaller cardinality than α and must therefore be bounded above in (regular) α. Select an upper bound u for it, notice that S_β cannot be bounded in α, and choose

x_β= the least element of S_β strictly greater than u'.

Now that induction guarantees the existence of the required $(x_\beta, \beta < \alpha)$, we observe that the set of successors of its terms

$$T = \{x'_\beta : \beta < \alpha\}$$

includes none of the x_β, and thus contains none of the S_β.

Lemma B. *Suppose that for cardinals α and β we have*

$$\alpha \leq 2^\beta \text{ and}$$

$$\gamma < \beta \text{ implies } 2^\gamma \leq \beta.$$

Then there is a Hausdorff topology on a set of cardinality α in which every subspace has a dense subset of cardinality β or less.

Proof: We first consider the power set $P(\beta)$ of β. For each $\gamma < \beta$ and each subset G of γ put

$$[\gamma, G] = \{H \in P(\beta) \ : \ H \cap \gamma = G\}.$$

Whenever $\gamma_1 \leq \gamma_2$ in β we see that

$$[\gamma_1, G_1] \cap [\gamma_2, G_2] = \begin{cases} [\gamma_2, G_2] & \text{if } G_2 \cap \gamma_1 = G_1, \\ \phi & \text{otherwise} \end{cases}$$

and it follows that $\mathcal{B} = \{[\gamma, G] \ : \ \gamma < \beta, G \subseteq \gamma\}$ is a base for a topology τ on $P(\beta)$. Given distinct elements A, B of $P(\beta)$ it will always be possible to find $\gamma < \beta$ such that γ belongs to exactly one of A and B; then $[\gamma', A\cap\gamma']$ and $[\gamma', B\cap\gamma']$ are disjoint τ-open neighbourhoods of A and B, so τ is Hausdorff. The cardinality of $\mathcal{B}$ is $\sum_{\gamma<\beta} 2^\gamma$ which, under the stated supposition, is β. Every subspace of $P(\beta)$ therefore has a base (and consequently a dense subset) with at most β members. Now any set of cardinality α can be injected into $P(\beta)$ to inherit a topology with the same property.

Lemma C. *Suppose that X is a Hausdorff space of regular cardinality α whose subspaces all have dense subsets of β or fewer points, and that $\alpha^\beta = \alpha$. Let there be given a family $\{S_\gamma : \gamma < \alpha\}$ of α subsets of X each possessing α elements. Then every α-element subspace Y of X has an α-element subspace into which none of the S_γ can be homeomorphically embedded.*

Proof: Fix Y. For each $\gamma < \alpha$, any homeomorphism h from S_γ onto a subset of Y is completely determined by its values on a dense subset of S_γ; since this subset may be taken to comprise no more than β elements, there cannot be more than $\alpha^\beta = \alpha$ of these homeomorphisms, for which reason the number of subsets of Y that are homeomorphic to any of the various S_γ is at most $\alpha \cdot \alpha = \alpha$. Lemma A now assures us that Y has an α-element subset containing no homeomorph of any S_γ.

Notice that if we put $S_\gamma = Y$ for every $\gamma < \alpha$ in Lemma C, it tells us that Y is not wqm. Consider now the following composite assertion $Q\min(\alpha)$ concerning an uncountable cardinal α:

(a) neither $\mathcal{T}(\alpha)$ nor $T_2 \cap \mathcal{T}(\alpha)$ is supported by its wqm members, and

(b) any subfamily of $\mathcal{T}(\alpha)$ or of $T_2 \cap \mathcal{T}(\alpha)$ which does support the whole
family must have more than α members .... $[Q\min(\alpha)]$

Our three lemmas now permit us to probe the relationship between set-theoretic axioms and the values of α for which this is a valid conclusion, thus:

Theorem.
(i) The assumption $\aleph_1^{\aleph_0} = \aleph_1$ gives us $Q\min(\aleph_1)$.
(ii) If $\mathfrak{c} = 2^{\aleph_0}$ is regular [note: this is a consequence of Martin's Axiom (MA), see [4, p.284]] then we get $Q\min(\mathfrak{c})$.
(iii) The continuum hypothesis $CH(2^{\aleph_0} = \aleph_1)$ implies $Q\min(\aleph_1)$.
(iv) If the generalized continuum hypothesis $GCH(2^\alpha$ is the successor of α for each $\alpha \geq \aleph_0)$ is assumed, we get $Q\min(\alpha)$ for every successor cardinal α.

Proof: Both (i) and (ii) follow directly from Lemma C without recourse to Lemma B, since the real line (or an $\aleph_1$-element subset of it) will suffice for the space X, choosing $\beta = \aleph_0$. (iii) follows immediately from (ii). If we assume GCH, then every successor cardinal α is of the form 2^β (where β is its immediate predecessor) and Lemma B supplies the Hausdorff space needed by Lemma C; also $\alpha^\beta = (2^\beta)^\beta = \alpha$ to complete the evidence for (iv).

Corollary (to Lemma C). *CH (or MA) implies that the real line contains no Toronto space (sqm uncountable non-discrete Hausdorff space).*

Application to Bankston's "Anti-".

Paul Bankston [1] developed a procedure, based on the connected/totally disconnected relationship, for converting any given topological invariant P into another, "anti-P": a space X is *anti-P* when the only subspaces Y of X that are P are those for which *every* topology on a set of Y's cardinality is P. A comprehensive survey of this topic up to 1989 will be found in an earlier issue of this Bulletin [7], to which we refer the reader for details.

Suppose now that P is a given hereditary (non-universal) property and that λ_P denotes the smallest cardinality of the non-P spaces. Matier and McMaster have identified the circumstances in which there is a hereditary invariant Q satisfying anti-$Q = P$ (a *hereditary pre-anti* for P) [8,9] and in the case $\lambda_P = \aleph_0$ they use the Ginsburg and Sands theorem to identify amongst these properties Q one which is logically strongest. The question they leave unresolved, of whether it is possible to do this also when $\lambda_P > \aleph_0$, will now be shown to depend on the existence of 'enough' wqm spaces of cardinality λ_P.

Let us first examine the special case in which no space of cardinality λ_P or more is P. Topologically this is of extreme triviality, since P then is the property of having fewer than λ_P points (such an invariant has been referred to as *cardinally decisive!*) but it turns out to provide an adequate illustration of techniques and results so far obtained.

Lemma D. *When P is cardinally decisive, then Q is a hereditary pre-anti for P if and only if*
(i) Q is hereditary,
(ii) $\lambda_Q = \lambda_P$,
(iii) $\mathcal{T}(\lambda_P)$ is supported by $Q \cap \mathcal{T}(\lambda_P)$.

Proof: Almost immediate from the definitions.

Proposition. *A cardinally decisive property P possesses a strongest hereditary pre-anti if and only if $\mathcal{T}(\lambda_P)$ is supported by its wqm members.*

Proof: Supposing that $\mathcal{T}(\lambda_P)$ is so supported, we define S to comprise all spaces on fewer than λ_P points together with all wqm spaces on exactly λ_P points. Using Lemma D, this is easily checked to be one of the hereditary pre-antis for P. If Q is any of the latter properties and X is wqm in $\mathcal{T}(\lambda_P)$ then Y sub X for some Q space Y in $\mathcal{T}(\lambda_P)$, and consequently X sub Y also, which shows X to be Q; hence every S space is Q, so S is indeed the strongest such invariant.

Conversely, suppose that there is a space X in $\mathcal{T}(\lambda_P)$ none of whose equicardinal subspaces is wqm. Given any hereditary pre-anti Q for P, there must be a space Y in $Q \cap \mathcal{T}(\lambda_P)$ such that Y sub X, and since Y is not wqm we can find Z in $\mathcal{T}(\lambda_P)$ with Z sub Y but not (Y sub Z). We define:

$$Q^* = \{T \in Q : \text{ not } (Y \text{ sub } T)\}.$$

Another appeal to Lemma D readily shows Q^* to be a hereditary pre-anti for P which, since it excludes the Q space Y, is strictly stronger than Q. We conclude that no strongest hereditary pre-anti for P can exist.

Corollary.
(i) [9] The strongest hereditary pre-anti for the class of finite spaces comprises the five GS spaces together with all the finite spaces.
(ii) Assuming CH, the class of countable spaces has no strongest hereditary pre-anti.
(iii) Assuming GCH, for each successor cardinal α the class of spaces having cardinalities less than α has no strongest hereditary pre-anti.

No very radical transformation of the argument above is needed to generalize from the cardinally decisive case to that in which some P spaces have λ_P or more elements. We obtain the following conclusions:

Lemma E.
(i) A wqm space of cardinality λ_P is anti-P if and only if it is non-P.
(ii) If X sub Y where Y is wqm and X has the same cardinality as Y, then X is wqm.

Theorem. *Let P be a topological invariant which possesses hereditary pre-antis. There exists a strongest hereditary pre-anti for P if and only if the class of non-P spaces in $\mathcal{T}(\lambda_P)$ is supported by its wqm members. When it exists, it consists of the wqm non-P spaces of cardinality λ_P together with all spaces of smaller cardinality.*

Amongst the questions so far unresolved in our investigations of this topic, the following appear to be most pressing:
Problem 1. Find a wqm space which is not sqm. More generally, for which values of α are wqm and sqm in $\mathcal{T}(\alpha)$ equivalent?
Problem 2. Will any reasonable set-theoretic assumptions enable us to prove or disprove $Q\min(\alpha)$ where α is an uncountable limit cardinal?

References

[1] P. Bankston, *The total negation of a topological property*, Illinois J. Math. **23** (1979), 241–252.

[2] J. Ginsburg and B. Sands, *Minimal infinite topological spaces*, Amer. Math. Monthly **86** (1979), 574–576.

[3] R. E. Larson, *Minimal T_0-spaces and minimal T_D-spaces*, Pacific J. Math. **31** (1969), 451–458.

[4] A. Levy, Basic Set Theory. Springer: Berlin-Heidelberg-New York, 1979.

[5] B. Johnston and S. D. McCartan, *Minimal T_F-spaces and minimal T_{FF}-spaces*, Proc. Royal Irish Acad. **80A** (1980), 93–96.

[6] B. Johnston and S. D. McCartan, *Minimal T_{YS}-spaces and minimal T_{DD}-spaces*, Proc. Royal Irish Acad. **88A** (1988), 23–28.

[7] J. Matier and T. B. M. McMaster, *Total negation in general topology*, Irish Math. Soc. Bull. **25** (1990), 26–37.

[8] J. Matier and T. B. M. McMaster, *Total negations of hereditary properties*, Rend. Circ. Mat. Palermo (2) **29** (1992), 563–573.

[9] J. Matier and T. B. M. McMaster, *Minimal countable spaces and anti-properties*, Rend. Circ. Mat. Palermo (2) **29** (1992), 553–561.

[10] A. E. McCluskey, Minimality with respect to separation axioms and sobriety, including their induced order characterizations, (Ph.D. thesis). Queen's University Belfast, 1990.

[11] A. E. McCluskey and S. D. McCartan, *Minimal sober spaces* (submitted).

[12] A. E. McCluskey and S. D. McCartan, *The minimal structures for T_A*, Ann. New York Acad. Sci. **659** (1992), 138–155.

[13] A. E. McCluskey and S. D. McCartan, *Minimality with respect to Young's axiom* (submitted).

[14] J. van Mill and G. M. Reed (eds.), Open problems in topology. North-Holland: Amsterdam-New York-Oxford-Tokyo, 1990.

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EPIMORPHISMS ACTING ON BURNSIDE

Des MacHale and Robert Sheehy

The Burnside group $B(r,n)$ is the group of exponent n, generated by r elements $x_1, x_2, \ldots, x_r$. It is well known that $B(r,n)$ is finite for $n = 2$, 3, 4 and 6 for all r but that for $n \geq 665$ and n odd, $B(r,n)$ is infinite when $r > 1$. In addition, it has recently been shown that for $n \geq 2^{48}$, $B(r,n)$ is infinite for $r > 1$, [1].

Let $\mathcal{B}$ be the set of all positive integers n for which $B(r,n)$ is finite for all r. Since the relation $g^n = 1$ can be written as $g^{n+1} = g = (g)I$ where I is the identity automorphism, we ask the following question.

Suppose G is a finitely generated group and the map α given by $g\alpha = g^k$ for all $g \in G$ and a fixed positive integer k, is an automorphism of G. What values of k force G to be finite?

In fact, in what follows, we can replace 'automorphism' by 'epimorphism', that is, an endomorphism of G onto G, and prove the following result.

Theorem. *Suppose that n belongs to $\mathcal{B}$ and that G is a finitely generated group such that the map α given by $g\alpha = g^{n+1}$ for all $g \in G$ is an epimorphism of G. Then G is finite.*

Proof: For all a and b in G, $(ab)\alpha = (ab)^{n+1} = a^{n+1}b^{n+1}$, so by cancellation $(ba)^n = a^n b^n$. Then $(ba)^{n+1} = (ba)^n ba = a^n b^n ba$, whence $b^{n+1}a^n = a^n b^{n+1}$. Since α is onto, $ga^n = a^n g$ for all a and g in G, and so $a^n \in Z(G)$ for all $a \in G$, where $Z(G)$ denotes the centre of G.

Now $G/Z(G)$, being a factor group of a finitely generated group, is finitely generated of exponent n and since $n \in \mathcal{B}$, $G/Z(G)$ is finite. Thus $Z(G)$, being a subgroup of finite index in a finitely generated group, is a finitely generated abelian group.

If $Z(G)$ is finite we are finished, so assume that $Z(G)$ is infinite. Then

$$Z(G) \simeq T \times C_\infty \times \cdots \times C_\infty$$

is the direct product of a finite group T and finitely many infinite cyclic groups. Now $Z(G)$ is invariant under all epimorphisms of G onto G, but clearly $\alpha : x \to x^{n+1}$ is not onto on any of the infinite cyclic factors. This contradiction establishes the result. ∎

Reference

[1] S. V. Ivanov, *On the Burnside problem on periodic groups*, Bull. Amer. Math. Soc. **27** (1992), 257–260.

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THE POISSON KERNEL AS AN EXTREMAL FUNCTION

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It has long been known that for certain classical inequalities involving positive harmonic functions on the open unit ball B of $\mathbf{R}^N$ the Poisson kernel of B (with some fixed pole on ∂B) is extremal (that is, a function for which equality holds in the inequalities). In recent years several new inequalities for positive harmonic functions on B have been discovered; again the Poisson kernel and functions related to it appear in extremal roles. This article, which is based on part of a talk given at the Society's Meeting at Waterford in September 1992, surveys some such inequalities, both old and new.

1. Harmonic functions and the Poisson kernel

1.1. *Harmonic functions*
A real-valued function h is *harmonic* on a non-empty open subset Ω of the Euclidean space $\mathbf{R}^N$, where $N \geq 2$, if h is smooth (that is, $h \in C^2(\Omega)$) and satisfies Laplace's equation (that is, $\Delta h \equiv 0$ on Ω, where $\Delta = \partial^2/\partial x_1^2 + \ldots + \partial^2/\partial x_N^2$). Harmonic functions are also characterized by Gauss' mean value property: h is harmonic on Ω if and only if h is continuous on Ω and, for each closed ball $\beta \subset \Omega$, the value of h at the centre of β is equal to its average value over the boundary $\partial\beta$ of β (see, e.g., Hayman and Kennedy [13, §1.5.5]).

For ease of reference, we list the classes of harmonic functions that we shall consider:

$$\begin{aligned} H_N &= \{h : h \text{ is harmonic on } B\}, \\ H_N^+ &= \{h \in H_N : h > 0 \text{ on } B, h(0) = 1\}, \\ HH_{m,N} &= \{h : h \text{ is a homogeneous harmonic polynomial} \\ &\qquad \text{of degree } m \text{ on } \mathbf{R}^N\}, \\ H_{m,N} &= \{h : h \text{ is a harmonic polynomial of degree at} \\ &\qquad \text{most } m \text{ on } \mathbf{R}^N\}, \\ H_{m,N}^+ &= \{h \in H_{m,N} : h > 0 \text{ on } B, h(0) = 1\}. \end{aligned}$$

Clearly the spaces $H_N, HH_{m,N}$ and $H_{m,N}$ are real vector spaces ($0 \in HH_{m,N}$ by convention). The normalization $h(0) = 1$ in the definitions of H_N^+ and $H_{m,N}^+$ is convenient and involves no real loss of generality.

1.2 *The Poisson kernel*
The Poisson integral and hence implicitly the Poisson kernel for B were introduced in the 1820's (Poisson [16], [17]) for $N = 2, 3$ in a construction aimed at solving (what later became known as) the Dirichlet problem for B. The *Poisson kernel* K of B is defined on $B \times \partial B$ by

$$K(x,y) = (1 - \|x\|^2)\|x - y\|^{-N}, \tag{1.2.1}$$

where $\|\cdot\|$ is the Euclidean norm on $\mathbf{R}^N$. A calculation (see, e.g., [13, p.32]) shows that if $y \in \partial B$ is fixed, then $K(\cdot, y) \in H_N^+$. If μ is a finite signed measure on ∂B, then the Poisson integral P_μ of μ is defined on B by

$$P_\mu(x) = \int_{\partial B} K(x,y)d\mu(y). \tag{1.2.2}$$

Passing the operator Δ under the integral sign in (1.2.2), we deduce from the harmonicity of the functions $K(\cdot, y)$ that $P_\mu \in H_N$; if, further, μ is a probability measure on ∂B (that

is $\mu \geq 0$ and $\mu(\partial B) = 1$), then $P_\mu \in H_N^+$. The importance of the Poisson integral lies partly in the converse result which, following Doob [7, 1.II.4], we call the *Riesz-Herglotz representation theorem:*
if $h \in H_N^+$, then $h = P_\mu$ for some probability measure μ on ∂B.
(A proof can also be found, e.g., in Helms [14, Theorem 2.13].)

We explain in passing the connection between Poisson integrals and the Dirichlet problem, mentioned above. If $\mu = f\sigma$, where $f : \partial B \to \mathbf{R}$ is continuous on ∂B and σ is surface measure on ∂B normalized so that $\sigma(\partial B) = 1$, then $P_\mu \in H_N$ and $P_\mu(x) \to f(y)$ as $x \to y$ for all $y \in \partial B$; that is to say, P_μ solves the classical Dirichlet problem for B with boundary function f.

Our aim here is to illustrate the extremal role of the Poisson kernel in relation to inequalities involving three classes of functions.

2. Inequalities for positive harmonic functions

We mention three inequalities (two classical, one recent) for functions of class H_N^+.

2.1 *Harnack's inequalities*
It is easy to see that

$$(1-\|x\|)(1+\|x\|)^{1-N} \leq K(x,y) \leq (1+\|x\|)(1-\|x\|)^{1-N} \tag{2.1.1}$$

for all $x \in B$ and all $y \in \partial B$. If $h \in H_N^+$, then, by the Riesz-Herglotz theorem, $h = P_\mu$ for some probability measure μ on ∂B. Integrating each member of (2.1.1) with respect to $d\mu(y)$, we obtain the Harnack inequalities [12]:
For $h \in H_N^+, x \in B$,

$$(1 - \|x\|)(1 + \|x\|)^{1-N} \leq h(x) \leq (1 + \|x\|)(1 - \|x\|)^{1-N}. \tag{2.1.2}$$

For (2.1.2) the Poisson kernel is extremal: more precisely, examining cases of equality in (2.1.1), we find that if $x \in B\backslash\{0\}$, the left-hand (respectively, right-hand) inequality in (2.1.2) is strict unless $h = K(\cdot, y)$ for some $y \in \partial B$ and $x = -\alpha y$ (respectively, $x = \alpha y$) for some $\alpha \in (0, 1)$.

2.2 *A corollary of Harnack's inequalities*
From (2.1.2) it follows easily that if $h \in H_N^+$, then

$$||x||^{-1}|h(x) - h(0)| \le N + O(||x||) \qquad \text{as } ||x|| \to 0,$$

whence

$$||\nabla h(0)|| \le N \qquad (h \in H_N^+), \tag{2.2.1}$$

where ∇ is the gradient operator: $\nabla = (\partial/\partial x_1, \ldots, \partial/\partial x_N)$. In particular,

$$|(\partial h/\partial x_1)(0)| \le N \qquad (h \in H_N^+). \tag{2.2.2}$$

Calculations show that equality holds in (2.2.1) if $h \in K(\cdot, y)$ for some $y \in \partial B$ and in (2.2.2) if $h = K(\cdot, y)$ with $y = (\pm 1, 0, \ldots, 0)$; with a little more trouble one can show that these are the only cases of equality.

2.3 *A generalization of* (2.2.2.)
Goldstein and Kuran [10] generalized (2.2.1) and (2.2.2). As a sample of their work, we state a generalization of (2.2.2) in the case $N = 3$: for $m = 1, 2, \ldots$

$$-m!(2m+1) \le (\partial^m h/\partial x_1^m)(0) \le m!(2m+1) \quad (h \in H_3^+); \tag{2.3.1}$$

moreover, equality holds on the right hand side if and only if $h = K(\cdot, (1, 0, 0))$. (The question of sharpness in the left-hand inequality is another story; see [3].) A proof of (2.3.1) is out of the question here, but the idea is to write h as a Poisson integral, pass the operator $\partial^m/\partial x_1^m$ under the integral sign in (1.2.2), and then (the hard part) estimate $(\partial^m/\partial x_1^m)K(x, y)$ at $x = 0$. In §4.1 we indicate the significance of the factor $2m + 1$ in (2.3.1).

3. Inequalities for trigonometric polynomials

3.1 *Trigonometric polynomials and harmonic polynomials*
To study trigonometric polynomials is essentially to study plane harmonic polynomials, as we now explain. We identify $\mathbf{R}^2$ with $\mathbf{C}$ in the usual way. For a positive integer j, the functions $\mathrm{Re}(z^j)$ and $\mathrm{Im}(z^j)$ span $HH_{j,2}$. Thus a typical element h of $H_{m,2}$ has the form

$$h(re^{i\theta}) = a_0 + \sum_{j=1}^{m} r^j (a_j \cos j\theta + b_j \sin j\theta), \tag{3.1.1}$$

where the coefficients are real. Let T_m denote the space of real-valued trigonometric polynomials (defined on the unit circle) of degree at most m. A typical element f of T_m can be written as

$$f(e^{i\theta}) = a_0 + \sum_{j=1}^{m} (a_j \cos j\theta + b_j \sin j\theta). \tag{3.1.2}$$

The isomorphism $\Phi : T_m \to H_{m,2}$ mapping the function in (3.1.2) to that in (3.1.1) clearly maps each $f \in T_m$ to the solution of the Dirichlet problem in the unit disc D with boundary function f. Let

$$T_m^+ = \{f \in T_m : f \ge 0 \text{ on } \partial D, \int_0^{2\pi} f(e^{i\theta})\, d\theta = 2\pi\}.$$

Note that the elements of $H_{m,2}^+$ and T_m^+ are normalized so that $a_0 = 1$ in the representations (3.1.1) and (3.1.2) respectively. It follows from the well-known minimum principle for harmonic functions that $\Phi(T_m^+) = H_{m,2}^+$. Thus results for T_m^+ can be interpreted for $H_{m,2}^+$.

3.2 *An inequality of Fejér*
Fejér [8] proved that

$$\sup_{\partial D} f \le m + 1 \qquad (f \in T_m^+). \tag{3.2.1}$$

Here is a quick proof. Write $f \in T_m^+$ in the form (3.1.2). Since $f \geq 0$ on ∂D and $a_0 = 1$, for all θ we have

$$\begin{aligned} f(e^{i\theta}) &\leq \sum_{k=0}^{m} f(e^{i(\theta+2k\pi/(m+1))}) \qquad (3.2.2)\\ &= m+1+\\ &\sum_{j=1}^{m}\sum_{k=0}^{m}\left\{a_j \cos\left(j\theta+\frac{2jk\pi}{m+1}\right)+b_j \sin\left(j\theta+\frac{2jk\pi}{m+1}\right)\right\}\\ &= m+1; \end{aligned}$$

the last step uses the equation

$$\sum_{k=0}^{m} e^{2ijk\pi/(m+1)} = 0 \quad (j = 1, \ldots, m).$$

It is easy to see that in working with $f \in T_m^+$, there is no real loss of generality in supposing that $\sup_{\partial D} f = f(1)$. It was also shown by Fejér that there is exactly one $f \in T_m^+$ such that $\sup_{\partial D} f = f(1) = m+1$, and this function is

$$f_m(e^{i\theta}) = 1 + 2(m+1)^{-1}\sum_{j=1}^{m}(m+1-j)\cos j\theta.$$

(A proof can be based on the observations that equality holds in (3.2.2) with $\theta = 0$ if and only if $f(e^{2ik\pi/(m+1)}) = 0$ for all $k = 1, \ldots, m$ and that f_m has this property, since

$$f_m(e^{i\theta}) = (m+1)^{-1}\sin^2((m+1)\theta/2)/\sin^2(\theta/2) \qquad (0 < \theta < 2\pi).)$$

Interpreted for harmonic polynomials, these results say that

$$\sup_{\partial D} h \leq m+1 \qquad (h \in H_{m,2}^+) \qquad (3.2.3)$$

and

$$h_m(re^{i\theta}) = 1 + 2(m+1)^{-1}\sum_{j=1}^{m}(m+1-j)r^j \cos j\theta \qquad (3.2.4)$$

is the only element of $H_{m,2}^+$ for which $\sup_{\partial D} h = h(1) = m+1$. To see how the extremal functions h_m are related to the Poisson kernel of D, note that writing $N = 2, x = re^{i\theta}, y = 1$ in (1.2.1) gives

$$K(re^{i\theta}, 1) = (1-r^2)(1+r^2-2r\cos\theta)^{-1} = 1 + \sum_{j=1}^{\infty} 2r^j \cos j\theta \qquad (3.2.5)$$

and that $h_m(re^{i\theta})$ is the mth Cesàro (that is, $(C,1)$) mean of this series (including the term 1); in particular, $h_m \to K(\cdot, 1)$ locally uniformly on D as $m \to \infty$.

3.3 *An inequality of Szegö*

If $f \in T_m^+$ is given by (3.1.2) (so that $a_0 = 1$), how big can the individual coefficients a_j, b_j be? For simplicity consider only a_j. A crude estimate is easy:

$$0 \leq \pi^{-1}\int_0^{2\pi}(1 \pm \cos j\theta) f(e^{i\theta})d\theta = 2 \pm a_j,$$

so that $|a_j| \leq 2$. Szegö's sharp result [18, p.625] is

$$|a_j| \leq 2\cos(\pi/(2+[m/j])), \qquad (3.3.1)$$

where $[\cdot]$ is the integer part function. Much later Kuran and I (unpublished) rediscovered the harmonic polynomial version : if $h \in H_{m,2}^+$ and h is given by (3.1.1), then a_j satisfies (3.3.1); note that $j!a_j = (\partial^j h/\partial x_1^j)(0)$. We were also able to say something about extremal functions. For example, with $j = 1$:

$$|(\partial h/\partial x_1)(0)| \leq 2\cos(\pi/(2+m)) \qquad (h \in H_{m,2}^+) \qquad (3.3.2)$$

(compare (2.2.2) with $N = 2$), and there is exactly one function $h_m \in H^+_{m,2}$ for which equality holds in (3.3.2) with the modulus sign suppressed; further $h_m \to K(\cdot, 1)$ locally uniformly on D as $m \to \infty$.

3.4 *A recent result and an open question*
A question of Holland [1, Problem 4.26] essentially asks for the value of

$$\Lambda_m = \sup_{f\in T^+_m} \frac{1}{2\pi}\int_0^{2\pi} (f(e^{i\theta}))^2\,d\theta = \sup_{h\in H^+_{m,2}} \frac{1}{2\pi}\int_0^{2\pi} (h(e^{i\theta}))^2\,d\theta.$$

No general formula for Λ_m has been found, but it is known that the suprema are attained (Goldstein and McDonald [11])and that $\lim \Lambda_m/m$ exists and equals 0.68698... (Garsia et al. [9]; see also Brown et al. [6]). It would be interesting to know whether the extremal functions are again related to the Poisson kernel of D: if $h_m \in H^+_{m,2}$ is such that $\sup_{\partial D} h_m = 1$ and

$$\Lambda_m = \frac{1}{2\pi}\int_0^{2\pi} (h_m(e^{i\theta}))^2\,d\theta,$$

do we have $h_m \to K(\cdot, 1)$ on D as $m \to \infty$? The conjecture that (Λ_m/m) is decreasing also seems to be open.

4. Inequalities for harmonic polynomials

4.1 *General remarks about harmonic polynomials*
In their guise as questions about harmonic polynomials, the problems discussed in §3 can be posed in all dimensions. They are generally more difficult in higher dimensions, since complex variable techniques are not readily available and representations of harmonic polynomials on $\mathbf{R}^N$ are not as simple as (3.1.1) when $N \geq 3$.

We need to quote some well-known facts; Brelot and Choquet [5] give an excellent account of most of these. Let u denote the unit vector $(1, 0, \ldots, 0)$ in $\mathbf{R}^N$. There is exactly one element $I_{m,N}$ of $HH_{m,N}$ such that $I_{m,N}$ is x_1-axial (that is, $I_{m,N}$ depends only on x_1 and $||x||$) and $I_{m,N}(u) = 1$. Writing $\cos\theta = x_1/||x||$ when $x \in \mathbf{R}^N\backslash\{0\}$, we have, for example,

$$I_{m,2}(x) = ||x||^m \cos m\theta, \quad I_{m,3} = ||x||^m P_m(\cos\theta),$$

where P_m is the Legendre polynomial of degree m. Also, we have $|I_{m,N}| \leq 1$ on ∂B and

$$\int_{\partial B} I^2_{m,N}\,d\sigma = 1/d_{m,N}, \tag{4.1.1}$$

where

$$d_{m,N} = \dim HH_{m,N} = \frac{2m+N-2}{m+N-2}\binom{m+N-2}{N-2}, \tag{4.1.2}$$

so that for $m \geq 1$

$$d_{m,2} = 2, \quad d_{m,3} = 2m+1, \quad d_{m,4} = (m+1)^2.$$

The N-dimensional generalization of (3.2.5) is

$$K(x,u) = 1 + \sum_{j=1}^{\infty} d_{j,N} I_{j,N}(x) \qquad (x \in B) \tag{4.1.3}$$

(see Müller [15, Lemma 17]). It is the appearance of the coefficients $d_{j,N} (= 2j+1$ when $N = 3)$ in (4.1.3) that ultimately accounts for the factor $2m+1$ in (2.3.1).

4.2 *An axialization technique*
We explain a device which greatly simplifies some extremal problems for harmonic polynomials. If $f : \mathbf{R}^N \to \mathbf{R}$ is continuous, then we define its x_1-*axialization* $f^* : \mathbf{R}^N \to \mathbf{R}$ by writing $f^*(x) = f(x)$ if x lies on the x_1-axis and otherwise defining $f^*(x)$ to be the average value of f on the set $\{y : y_1 = x_1, ||y|| = ||x||\}$. It turns out that if h is harmonic, then so also is h^*; moreover if $h \in HH_{m,N}$, then $h^* = h(u)I_{m,N}$ (see [2, Lemma] for details). This observation

allows us to reduce the proofs of several inequalities for harmonic polynomials to consideration of x_1-axial harmonic polynomials.

4.3 *A generalization of (3.2.3)*
Kuran and I [4] obtained explicit, best possible constants $C_{m,N}$ such that

$$\sup_{\partial B} h \leq C_{m,N} \qquad (h \in H^+_{m,N}).$$

We have $C_{m,2} = m+1$ (see (3.2.3)) and

$$C_{m,N} \sim 2^{2-N}((N-1)!)^{-1}m^{N-1}$$

as $m \to +\infty$ with N fixed. Our proof depends on the technique of §4.2 and known quadrature formulae for ultraspherical (Legendre when $N=3$) polynomials. As in the plane case (§3.2), there is a unique extremal element h_m of $H^+_{m,N}$ which satisfies the equation $\sup_{\partial B} h = h_m(u) = C_{m,N}$. We have no simple formula, corresponding to (3.2.4), for h_m when $N \geq 3$, but it remains true in all dimensions that $h_m \to K(\cdot, u)$ locally uniformly on B as $m \to \infty$.

4.4 *A generalization of (3.3.2)*
Szegö [18, p.626] obtained the following analogue of (3.3.2) for the case $N = 3$:

$$|(\partial h/\partial x_1)(0)| \leq 3\tau_m \qquad (h \in H^+_{m,3}), \tag{4.4.1}$$

where τ_m is the greatest zero of $P_{(m+2)/2}$ or $P_{(m+1)/2} + P_{(m+3)/2}$ according as m is even or odd. Note that $\tau_m \in [0,1)$ and $\tau_m \to 1$ as $m \to \infty$, and compare (4.4.1) with the case $N=3$ of (2.2.2). Techniques like those mentioned in §4.3 (axialization and quadrature formulae) can be used to generalize (4.4.1) to all dimensions. Again the extremal polynomials for the N-dimensional generalization of (4.4.1) are related to the Poisson kernel.

4.5 *Two norms on $H_{m,N}$*
For polynomials P, Q on $\mathbf{R}^N$, let

$$< P,Q > = \int_{\partial B} PQ \, d\sigma,$$
$$||P||_2 = \sqrt{< P,P >},$$
$$||P||_\infty = \sup_{\partial B} |P|.$$

Note that $< \cdot, \cdot >$ is not an inner product and $||\cdot||_2, ||\cdot||_\infty$ are not norms on the space of all such polynomials, for a polynomial may vanish on ∂B but be not identically zero. However $< \cdot,\cdot >$ is an inner product and $||\cdot||_2, ||\cdot||_\infty$ are norms on $H_{m,N}$. Clearly $||P||_2 \leq ||P||_\infty$ for all polynomials P. An inequality in the opposite direction for polynomials of degree at most m is

$$||P||_\infty \leq \sqrt{d_{m,N+1}}||P||_2, \tag{4.5.1}$$

and this is sharp. We briefly explain how the constant $\sqrt{d_{m,N+1}}$ comes to appear in (4.5.1). First note that $||P||_2$ and $||P||_\infty$ involve only the values of P on ∂B, and there is an element of $H_{m,N}$ which agrees with P on ∂B (see [5]). Hence we may suppose that $P \in H_{m,N}$. By a rotation of axes, we may further suppose that $||P||_\infty = |P(u)|$, and an argument based on the observations in §4.2 allows us to suppose that P is x_1-axial. Then P has the representation

$$P = a_0 I_{0,N} + a_1 I_{1,N} + \ldots + a_m I_{m,N},$$

so that

$$||P||_\infty = |P(u)| = |a_0 + \ldots + a_m|. \tag{4.5.2}$$

Further, by (4.1.1) and the orthogonality relation

$$< I_{j,N}, I_{k,N} > = 0 \qquad (0 \leq j < k),$$

we obtain

$$||P||_2 = \sqrt{\sum_{j=0}^{m} a_j^2/d_{j,N}}. \tag{4.5.3}$$

In view of (4.5.2) and (4.5.3), the Cauchy-Schwarz inequality gives

$$||P||_\infty \leq \sqrt{d_{0,N} + \ldots + d_{m,N}}||P||_2.$$

A calculation using (4.2.2) shows that $d_{0,N}+\ldots+d_{m,N} = d_{m,N+1}$, and (4.5.1) now follows.

Checking for cases of equality at each stage of the argument, we find without much difficulty that if $P \in H_{m,N}$ and equality holds in (4.5.1), then $P = \alpha K_m$ for some real α, where K_m is the mth partial sum of the series expansion (4.1.3) of $K(\cdot, u)$, that is

$$K_m = 1 + \sum_{j=1}^{m} d_{j,N} I_{j,N}.$$

It is hoped that details of (4.5.1) and some related inequalities will appear elsewhere.

References

[1] J. M. Anderson, K. F. Barth and D. A. Brannan, *Research problems in complex analysis*, Bull. London Math. Soc. **9** (1977), 129–162.

[2] D. H. Armitage, *Spherical extrema of harmonic polynomials*, J. London Math. Soc. (2) **19** (1979), 451–456.

[3] D. H. Armitage, *Polynomial expansions of positive harmonic functions in the unit ball*, Math. Proc. Camb. Phil. Soc. **110** (1991), 533–544.

[4] D. H. Armitage and Ü. Kuran, *Sharp bounds for harmonic polynomials*, J. London Math. Soc. (2) **42** (1990), 475–488.

[5] M. Brelot and G. Choquet, Polynômes harmoniques et polyharmoniques, (Seconde colloque sur les équations aux derivées partielles): Bruxelles, 1954.

[6] J. Brown, M. Goldstein and J. McDonald, *A sequence of extremal problems for trigonometric polynomials*, J. Math. Anal. Appl. **130** (1988), 545–551.

[7] J. L. Doob, Classical potential theory and its probabilistic counterpart. Springer, 1983.

[8] L. Fejér, *Sur les polynomes harmoniques quelconques*, C. R. Acad. Sci. Paris **157** (1913), 506–509.

[9] A. Garsia, E. Rodemich and H. Rumsey, *On some extremal positive definite functions*, J. Math. Mech. **18** (1969), 805–834.

[10] M. Goldstein and Ü. Kuran, *On Harnack-type inequalities and the extremality of the Poisson kernel*, Complex Variables **9** (1988), 327–342.

[11] M. Goldstein and J. N. McDonald, *An extremal problem for non-negative trigonometric polynomials*, J. London Math. Soc. (2) **29** (1984), 81–88.

[12] A. Harnack, *Existenzbeweise zur Theorie des Potentials in der Ebene und im Raume*, Leipziger Ber. (1886), 144–169.

[13] W. K. Hayman and P. B. Kennedy, Subharmonic functions (Vol. 1). Academic Press, 1976.

[14] L. L. Helms, Introduction to potential theory. Wiley, 1969.

[15] C. Müller, Spherical harmonics. Springer, 1966.

[16] S. D. Poisson, *Addition au mémoire précédent et au mémoire sur la manière d'exprimer les fonctions par les séries de quantités périodiques*, J. École Roy. Polytechnique (19) **12** (1823), 145–162.

[17] S. D. Poisson, *Mémoire sur le calcul numérique des intégrales définies*, Mémoires de l'acad. Royale des sci. de l'institut de France **6** (1823), 571–602.

[18] G. Szegö, *Koeffizientenabschäzungen bei ebenen und räumlichen harmonischen Entwicklungen*, Math. Ann. **96** (1927), 601–632.

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RECENT RESULTS ON THE ORDER STRUCTURE OF COMPACT OPERATORS

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Abstract: There have been only a few really positive results concerning the order structure of spaces of compact operators on Banach lattices, although many related open questions have been posed over the years. Recent results by the authors show why this is so—those few positive results describe virtually all that is true!

1 Introduction.

We will consider linear operators between real Banach lattices. A Banach lattice is a Banach space, E, which is also a vector lattice with the two structures related by the implication $|x| \leq |y| \Rightarrow \|x\| \leq \|y\|$ for all $x, y \in E$. If E and F are Banach lattices then we will be considering various subspaces of the spaces $\mathcal{L}(E,F)$ of bounded linear operators, and $\mathcal{K}(E,F)$ of compact linear operators.

If E and F are Banach lattices then an operator $T : E \to F$ is termed *positive* if $x \geq 0 \Rightarrow Tx \geq 0$. The linear span of the positive operators is the space of *regular* operators, denoted by $\mathcal{L}^r(E,F)$. Usually $\mathcal{L}^r(E,F)$ is a proper subspace of $\mathcal{L}(E,F)$. When we order $\mathcal{L}^r(E,F)$ by defining $S \geq T \Leftrightarrow S - T$ is positive, the space $\mathcal{L}^r(E,F)$ is certainly made into a partially ordered vector space, but is not, in general, a lattice. The most important case in which it is a lattice is when F is Dedekind complete, i.e. when every non-empty set with an upper bound must have a least upper bound (the corresponding assertion for lower bounds follows automatically). In this case $\mathcal{L}^r(E,F)$ is itself a Dedekind complete vector lattice. The basic results of the theories of Banach

lattices and of positive operators may be found in [8], [29] or [37] as well as in several other texts.

One reason for studying compact operators is that there seemed, at least for some years, a good chance that the order structure of, say, the compact regular operators might be rather better than that of regular operators. There are of course other reasons for their study. The work of Krengel that we describe in the next section had its origins in ergodic theory, whilst the Dodds-Fremlin theorem has applications in theoretical physics (see [11]). A reasonable hope might have been that the compact operators between two Banach lattices formed a lattice under the operator order. This turns out not to be true but slightly lower expectations would still seem to be reasonable. These hopes turned out to be forlorn in general. There are some partial positive results, for the statements of which we need to give a few definitions.

In this survey, we have concentrated solely on the order structure of spaces of compact operators. There are many other topics in the theory of compact positive operators which lie off the main direction chosen for this survey and on which substantial progress has been made in recent years. Topics that we could have mentioned include results in [10], [13], [14], [17], [18], [21], [35], [36] and [39] on factorizing compact positive operators; the vast literature on the spectral theory of compact positive operators; a special and important part of the latter stemming from the Andô-Krieger theorem and culminating in de Pagter's proof, in [33], that an irreducible compact positive operator has strictly positive spectral radius and in further refinements obtained in [3]; and many other areas.

2. Some Banach lattice terminology.

There are two special classes of Banach lattices that have been studied almost as long as Banach lattices themselves. An *AM-space* is a Banach lattice in which $\|x \vee y\| = \|x\| \vee \|y\|$ whenever $x, y \geq 0$. It was shown in [23] and in [25] that each AM-space is isometrically order isomorphic to a closed sublattice of some space $C(K)$ where K is a compact Hausdorff space. An *AL-space* is a

Banach lattice in which $\|x+y\| = \|x\| + \|y\|$ whenever $x, y \geq 0$. Again it was shown in [22] that each AL-space is isometrically order isomorphic to an $L^1(\mu)$-space for some measure μ.

The normed dual, E', of a Banach lattice E may be naturally ordered by defining $f \geq g \Leftrightarrow f(x) \geq g(x)$ for all $0 \leq x \in E$. Under this order E' is also a Banach lattice (and is even Dedekind complete). The concepts of AM- and AL-spaces are mutually dual, i.e. the dual of an AM-space is an AL-space whilst the dual of an AL-space is an AM-space.

A subspace J of a Banach lattice E is an *ideal* whenever $x \in E, y \in J$ and $|x| \leq |y|$ imply that $x \in J$. A *band* is an ideal with the extra property that if a subset of J has a supremum in E then that supremum must actually lie in J.

A Banach lattice has an *order continuous norm* if every downward directed family with infimum equal to zero must converge in norm to zero. An equivalent condition is that the Banach lattice be an ideal in its bidual. Banach lattices with an order continuous norm are Dedekind complete. If $1 \leq p < \infty$ then each space $L^p(\mu)$ has an order continuous norm. Spaces $C(K)$ have an order continuous norm only if K is a finite set.

A Banach lattice is a *KB-space* (=Kantorovich-Banach space) if it has an order continuous norm, and every norm-bounded upward directed set has a supremum. Various equivalences of this are known. One is that the Banach lattice is weakly sequentially complete, another is that it be a band in its bidual. All the spaces $L^p(\mu)$, for $1 \leq p < \infty$ are KB-spaces. The space of all null-sequences, c_0, with the supremum norm and the pointwise ordering, is an example of a Banach lattice which has an order continuous norm but which is not a KB-space.

An *atom* in a Banach lattice is a non-zero positive element e such that if $0 \leq x \leq e$ then x is a multiple of e. A Banach lattice is *atomic* if for every $0 < x \in E$ there is an atom e with $e \leq x$. $L^p(\mu)$ is atomic if and only if μ is a discrete measure, whilst $C(K)$ is atomic precisely when K has a dense subset of isolated points. Below, at some point we will meet the notion of atomic Banach lattices with an order continuous norm. Archetypal examples are ℓ_p, for $1 \leq p < \infty$ and c_0.

3. Krengel's results.

There are several possible questions that one might ask about the order structure of a subspace $\mathcal{I}$ of the space of bounded operators. We certainly want to know whether or not $\mathcal{I}$ is positively generated and whether or not it is a lattice. Furthermore if $\mathcal{I}$ is a lattice, then we want to know whether or not the lattice operations in $\mathcal{I}$ are also the corresponding lattice operations in the space of all regular operators. In general we cannot talk about $\mathcal{I}$ being a sublattice of the space of all regular operators as the latter space need not be a lattice (unless, for example, the range is Dedekind complete). There is no suitable terminology in the literature describing this situation; so it seems reasonable to extend the definition of sublattices as follows. If $\mathcal{J}$ is a partially ordered vector space and $\mathcal{I}$ a subspace of $\mathcal{J}$ then we say that $\mathcal{I}$ is a (*generalized*) *sublattice* of $\mathcal{J}$ if $\mathcal{I}$ is a lattice and for each $x, y \in \mathcal{I}$ the supremum of x and y calculated in $\mathcal{I}$ is also their supremum in $\mathcal{J}$. In the case when $\mathcal{J}$ is a vector lattice this is the usual definition of a sublattice. We will often omit the adjective "generalized" unless it is necessary to emphasize that the ambient space is not a lattice.

Using the properties of the compact subsets of $C(K)$-spaces, Krengel, in [26], established the first positive result in this area by proving the following.

Theorem 3.1. [Krengel] *If E is an arbitrary Banach lattice and F an arbitrary AM-space then $\mathcal{K}(E,F)$ is a generalized sublattice of $\mathcal{L}^r(E,F)$.*

In particular this means that every compact operator taking values in an AM-space does have a modulus (in $\mathcal{L}^r(E,F)$) and this modulus is again compact. A simple duality argument establishes a similar result if E is an AL-space and F is a KB-space. Later we shall see that this result can be improved somewhat.

In [27], Krengel gave two important examples, showing that these results are certainly not true in general. Much work in this area since then has been devoted to trying to rescue some remnant of his earlier positive results for spaces different from AM-spaces.

Example 3.2. [Krengel] *There is a Dedekind complete Banach lattice E and a compact operator T on E such that $|T|$ exists in $\mathcal{L}^r(E)$, but is not compact.*

The crucial feature of the corresponding construction is as follows. Consider a $2^n \times 2^n$ matrix with orthogonal rows and with all entries being ± 1. If this matrix is regarded as an operator S_n on 2^n-dimensional Hilbert space, E_n, then $\|S_n\| = 2^{n/2}$ whilst $\||S_n|\| = 2^n$. The required example is produced by taking a suitable weighted sum of the operators S_n acting on $\left(\sum E_n\right)_{c_0}$, the c_0-sum of the spaces E_n, whose elements are sequences (x_n) with $x_n \in E_n$ and $\|x_n\| \to 0$. All the examples subsequently produced in this field are based on modifications of various degrees of complexity of this construction.

This example shows already that even on Dedekind complete Banach lattices, the compact operators do not form a sublattice of the lattice of all regular operators. Although the operator T in Example 3.2 is a regular operator, it is easy to verify that T is not the difference of two compact positive operators, so that the space of compact operators on E is not positively generated. In fact a modification of the previous example shows that the compact operators are not even a subspace of $\mathcal{L}^r(E)$.

Example 3.3. [Krengel] *There is a Dedekind complete Banach lattice E and a compact operator T on E such that $|T|$ does not exist in $\mathcal{L}^r(E)$.*

Compact regular operators taking values in a Dedekind complete Banach lattice must have a modulus as all regular operators then do, so that the operator in Example 3.3 is not regular. Krengel's examples left open the possibility that for compact regular operators the condition of Dedekind completeness could be dropped. However, using Krengel's basic finite-dimensional building blocks, we constructed in [6] an example to show that this also was false.

Example 3.4. [Abramovich & Wickstead] *There is a Banach lattice E and a compact regular operator T on E such that $|T|$ does not exist in $\mathcal{L}^r(E)$.*

In the example that we constructed in [6] the operator T is *not* the difference of two positive compact operators. This led us to the believe that the *correct* space to study was the linear span of the positive compact operators rather than the space of all compact operators. We denote the space of differences of positive compact operators from E into F by $\mathcal{K}^r(E,F)$. Thus, $\mathcal{K}^r(E,F)$ is a subspace of $\mathcal{K}(E,F) \cap \mathcal{L}^r(E,F)$. By either looking at Krengel's proof of Example 3.2 or using its statement together with the Dodds-Fremlin theorem (Theorem 4.1 below), one can see that even when F is assumed to be Dedekind complete $\mathcal{K}^r(E,F) \neq \mathcal{K}(E,F) \cap \mathcal{L}^r(E,F)$. Although this space $\mathcal{K}^r$ must surely be better behaved than, say, the space of compact regular operators $\mathcal{K}(E,F) \cap \mathcal{L}^r(E,F)$, it does not behave all that well. In [7] we, again using Krengel's basic building blocks, produced two more examples.

Example 3.5. [Abramovich & Wickstead] *There is a Banach lattice E and $T \in \mathcal{K}^r(E)$ which does not have a modulus either in $\mathcal{K}^r(E)$ or in $\mathcal{L}^r(E)$.*

If E were Dedekind complete then operators in $\mathcal{K}^r(E)$ must certainly have a modulus in $\mathcal{L}^r(E)$, however...

Example 3.6. [Abramovich & Wickstead] *There is a Dedekind complete Banach lattice E and $T \in \mathcal{K}^r(E)$ which does not have a modulus in $\mathcal{K}^r(E)$.*

In particular the modulus of T, computed in $\mathcal{L}^r(E)$ is not compact.

There seems to be little left to conjecture as being true in great generality. How much further can we extend Krengel's positive results? His proof of Theorem 3.1 actually establishes that in that case $\mathcal{K}(E,F)$ is a Banach lattice under the operator order and the usual operator norm (there is another norm used in the study of regular operators, but we have no need of it in this survey). There are few cases where this holds that Krengel did not already deal with. The following theorem is due to Krengel [26], Cartwright and Lotz [12], and Schwarz [38, Theorem 8.1]. A simple proof of Schwarz's contribution will appear in [42].

Theorem 3.7. *If E and F are Banach lattices then $\mathcal{K}(E,F)$ is a Banach lattice under the operator order and norm if, and only if, either E is an AL-space or F is an AM-space.*

Certainly if either E is isomorphic to an AL-space or F is isomorphic to an AM-space then $\mathcal{K}(E,F)$ is isomorphic to a Banach lattice. However, there is no isomorphic version of Theorem 3.7. As pointed out in [2], it follows from an example in [1] and Theorem in [19] that the next result is true.

Example 3.8. *There are Dedekind complete Banach lattices E and F such that E is not isomorphic to an AL-space and F is not isomorphic to an AM-space but $\mathcal{K}(E,F)$ coincides with $\mathcal{K}^r(E,F)$ and is isomorphic to a Banach lattice.*

4. The Dodds-Fremlin theorem and its consequences.

Apart from Krengel's examples, little positive had been known about the order structure of spaces of compact operators until Dodds and Fremlin published their now celebrated theorem in [15]. In retrospect it is clear that there are many antecedents of this result in the literature, including [28], [31], [24, Theorem 5.10], [34] and [37, Theorem 10.2], but at the time the result came to most people as a complete surprise.

Theorem 4.1. [Dodds & Fremlin] *If E and F are Banach lattices such that both E' and F have order continuous norms, $T : E \to F$ is a positive compact linear operator and $S : E \to F$ is a linear operator such that $0 \leq S \leq T$ then S is compact.*

We refer to the conclusion of this theorem as the *compact domination property.* The Dodds-Fremlin condition (i.e. the continuity of norms in E' and F) is not the only one that guarantees the compact domination property. The other two, found in [40], are very strong conditions and the proofs of the compact domination property in these latter cases is rather simple. The conditions are that E' (resp. F) be atomic *and* have an order continuous norm. Some important, but easily deduced, consequences of the compact domination property are the following:

(α) If F is Dedekind complete (which is automatic if the Dodds-Fremlin condition holds) then $\mathcal{K}^r(E,F)$ is an (order) ideal in $\mathcal{L}^r(E,F)$ and therefore $\mathcal{K}^r(E,F)$ is a Dedekind complete vector lattice. In particular if $S,T : E \to F$ are two positive compact operators then $S \vee T$, which automatically exists in $\mathcal{L}^r(E,F)$, is compact and thus belongs to $\mathcal{K}^r(E,F)$.

(β) If F is only assumed to be Dedekind σ-complete then $\mathcal{K}^r(E,F)$ is a Dedekind σ-complete vector lattice. The only reason we cannot say that $\mathcal{K}^r(E,F)$ is an ideal in $\mathcal{L}^r(E,F)$ is that the latter space need not be a lattice.

Even in as apparently nice a context as that of operators into an AM-space, no analogue of the first conclusion in (α) is true, i.e. $\mathcal{K}^r(E,F)$ may easily fail to be Dedekind complete. To demonstrate this let E be an AL-space and X be a compact Hausdorff space. It has been known since [32] that $C(X)$ is Dedekind complete if and only if X is *Stonean,* i.e. the closure of every open subset of X is again open. There is an isometric order isomorphism between $\mathcal{K}(E,C(X))$ and $C(X,E')$. Since E' is a Dedekind complete AM-space, it can be identified with a space $C(Y)$ for some compact Hausdorff Stonean space Y. We may now identify $C(X,E')$ with $C(X,C(Y))$ and hence with $C(X \times Y)$ for both the norm and order structure. It follows from a well-known, but unpublished, result of W. Rudin (see [16] for a short proof using order theoretic notions) that $X \times Y$ is Stonean only when one factor is finite and the other Stonean. Thus as long as both E and $C(X)$ are infinite dimensional the space $K(E,C(X))$ cannot be Dedekind complete (or even Dedekind σ-complete).

There is an obvious interest in extending these known conditions which guarantee the compact domination property. At first sight it looks plausible that there is a whole spectrum of conditions that will do, with the two extremes being when either E' or F is atomic with an order continuous norm, and the Dodds-Fremlin theorem simply identifying an easily described case somewhere in the middle of the range. However, as recently shown in [41], that is not the case, and this makes the Dodds-Fremlin result all the more remarkable.

Theorem 4.2. [Wickstead] *The pair of Banach lattices E and F has the compact domination property if and only if one of the following three non-exclusive conditions holds:*

(a) Both E' and F have an order continuous norm.
(b) E' is atomic and has an order continuous norm.
(c) F is atomic and has an order continuous norm.

It is similarly surprising that the compact domination property is not just a simple sufficient condition for proving the two consequences (α) and (β) mentioned above. The following two results are proved in [42].

Theorem 4.3. [Wickstead] *If E and F are Banach lattices then $\mathcal{K}^r(E,F)$ is a Dedekind complete vector lattice if and only if the pair (E,F) has the compact domination property and F is Dedekind complete.*

Theorem 4.4. [Wickstead] *If E and F are Banach lattices then $\mathcal{K}^r(E,F)$ is a Dedekind σ-complete vector lattice if and only if the pair (E,F) has the compact domination property and F is Dedekind σ-complete.*

Notice now that the second conclusion in (α), that the supremum of two positive compact operators exists and is compact, is not an equivalence of the compact domination property. For example, by Theorem 3.1, this is also the case whenever F is an AM-space. Moreover, in all previously known cases the supremum of two positive compact operators was always compact whenever it existed, in particular whenever F was Dedekind complete. This led C. D. Aliprantis and O. Burkinshaw to ask for a counterexample to or a proof of this phenomenon. The question was posed by them at a *Riesz Spaces and Operator Theory* meeting at Oberwolfach in 1982, and reiterated in [20, Problem 6]. Unfortunately, the answer is negative; namely the examples in [6] show the following:

Example 4.5. [Abramovich & Wickstead] *There is a Dedekind complete Banach lattice E and compact positive operators $S,T : E \to E$ such that $S \vee T$ is not compact.*

Example 4.6. [Abramovich & Wickstead] *There are Banach lattices E and F and compact positive operators $S,T : E \to F$ such that $S \vee T$ does not exist in $\mathcal{K}^r(E,F)$.*

Before leaving this section we should mention, although the results are not directly related to the study of the order structure of spaces of compact operators, the extensions of the Dodds-Fremlin theorem proved by C. D. Aliprantis and O. Burkinshaw, in their remarkable work [9]. They managed to find some "hidden" compactness in positive operators dominated by a compact positive operator by proving the following theorem.

Theorem 4.7. [Aliprantis & Burkinshaw] *Let $0 \le S \le T$ be two positive operators on a Banach lattice E and assume that T is compact. Then operator S^3 is compact. If either E or E' has order continuous norm, then S^2 is compact.*

This theorem has many applications, of which the most interesting are in connection with the spectral properties of positive operators [3], [33] and with the invariant subspace problem for positive operators [4], [5]. For some applications in connection with positive semigroups we refer to [30] and references therein.

5. What is left to prove?

Although many conjectures have now been disposed of, there do remain some open questions in this area. We have characterized the cases in which $\mathcal{K}^r(E,F)$ is either a Dedekind complete or Dedekind σ-complete vector lattice. There is still no answer known to:

Question 5.1. *For what pairs of Banach lattices E and F is $\mathcal{K}^r(E,F)$ a vector lattice?*

The answer will certainly *not* be that the pair satisfy the compact domination property, since the conclusion also holds whenever F is an AM-space or E is an AL-space. On the other hand Example 4.6 shows that $\mathcal{K}^r(E,F)$ may fail to be a vector lattice. In all the cases that we do have a lattice, $\mathcal{K}^r(E,F)$ is a generalized sublattice of $\mathcal{L}^r(E,F)$. Possibly the following question might be rather more tractable.

Question 5.2. *For what pairs of Banach lattices E and F is $\mathcal{K}^r(E,F)$ a generalized sublattice of $\mathcal{L}^r(E,F)$?*

However both of these questions seem rather difficult at present. Perhaps the best that we might hope for will be an answer to:

Question 5.3. *If $\mathcal{K}^r(E,F)$ is a vector lattice, must it be a generalized sublattice of $\mathcal{L}^r(E,F)$?*

If it is so difficult for spaces of compact operators to have a lattice structure, is there some useful weaker order theoretic structure that we can look for? The *Riesz separation property* states that if $x_1, x_2 \leq z_1, z_2$ then there is y with $x_1, x_2 \leq y \leq z_1, z_2$. This condition is (slightly) weaker than that of being a lattice but has some important consequences. For example it, together with a fairly natural condition relating the norm and order, are equivalent to the dual of an ordered normed space being a Banach lattice under the dual ordering. In [43] the second author showed that it is possible to find Banach lattices E and F such that $\mathcal{L}^r(E,F)$ has the Riesz separation property, but is not a lattice. It is also possible to choose E and F such that $\mathcal{L}^r(E,F)$ does not have the Riesz separation property. The proofs used in [43] do not answer the corresponding questions for $\mathcal{K}^r(E,F)$, so these questions remain open. It is probably too much to expect an answer to:

Question 5.4. *For what pairs of Banach lattices E and F does $\mathcal{K}^r(E,F)$ have the Riesz separation property?*

But we would certainly hope for answers to the next two questions. In particular we feel that the answer to Question 5.6 is almost certainly positive.

Question 5.5. *Are there Banach lattices E and F such that $\mathcal{K}^r(E,F)$ has the Riesz separation property, but is not a lattice?*

Question 5.6. *Are there Banach lattices E and F such that $\mathcal{K}^r(E,F)$ does not have the Riesz separation property?*

References

[1] Y. A. Abramovich, *On the space of operators between Banach lattices*, Zapiski LOMI **73** (1977), 188–192.

[2] Y. A. Abramovich, *When each regular operator is continuous* in Functional Analysis, Optimization and Mathematical Economics, Oxford University Press: New York, 1990, 133–140.

[3] Y. A. Abramovich, C. D. Aliprantis and O. Burkinshaw, *On the spectral radius of positive operators*, Math. Z. **211** (1992), 593–607.

[4] Y. A. Abramovich, C. D. Aliprantis and O. Burkinshaw, *Invariant subspaces of positive operators*, J. Funct. Analysis (forthcoming).

[5] Y. A. Abramovich, C. D. Aliprantis and O. Burkinshaw, *Invariant subspaces for positive operators acting on a Banach space with basis*, Proc. Amer. Math. Soc. (forthcoming).

[6] Y. A. Abramovich and A. W. Wickstead, *A compact regular operator without modulus*, Proc. Amer. Math. Soc. **116** (1992), 721–726.

[7] Y. A. Abramovich and A. W. Wickstead, *Solutions of several problems in the theory of compact positive operators* (submitted).

[8] C. D. Aliprantis and O. Burkinshaw, Positive Operators. Academic Press: New York & London, 1985.

[9] C. D. Aliprantis and O. Burkinshaw, *Positive compact operators on Banach lattices*, Math. Z. **174** (1980), 289–298.

[10] C. D. Aliprantis and O. Burkinshaw, *Factoring compact and weakly compact operators through reflexive Banach lattices*, Trans. Amer. Math. Soc. **183** (1984), 369–381.

[11] J. Avron, I. Herbst and B. Simon, *Schrödinger operators with magnetic fields, I. General interactions*, Duke Math. J. **45** (1978), 847–883.

[12] D. I. Cartwright and H. P. Lotz, *Some characterizations of AM- and AL-spaces*, Math. Z. **142** (1975), 97–103.

[13] Z. L. Chen, *The factorization of a class of compact operators*, Dongbei Shuxue **6** (1990), 303–309.

[14] W. J. Davis, T. Figiel, W. B. Johnson and A. Pelczynski, *Factoring weakly compact operators*, J. Funct. Anal. **17** (1974), 311–327.

[15] P. G. Dodds and D. H. Fremlin, *Compact operators in Banach lattices*, Israel J. Math. **34** (1979), 287–320.

[16] Z. Ercan and A. W. Wickstead, *Banach lattices of continuous Banach lattice-valued functions* (submitted).

[17] T. Figiel, *Factorization of compact operators and applications to the approximation property*, Studia Math. **45** (1973), 191–210.

[18] N. Ghoussoub and W. B. Johnson, *Factoring operators through Banach lattices not containing* $C(0,1)$, Math. Z. **194** (1987), 153–171.

[19] G. Godefroy, *Sur les opérateurs compacts réguliers*, Bull. Soc. Belg. A **30** (1978), 35–37.

[20] C. B. Huijsmans and W. A. J. Luxemburg (eds.), Positive Operators and Semigroups on Banach Lattices. Kluwer: Dordrecht, 1992.

[21] W. B. Johnson, *Factoring compact operators*, Israel J. Math. **9** (1971), 337–345.

[22] S. Kakutani, *Concrete representations of abstract (L)-spaces and the mean ergodic theorem*, Ann. of Math. **42** (1941), 523–537.

[23] S. Kakutani, *Concrete representation of abstract (M)-spaces*, Ann. of Math. **42** (1941), 994–1024.

[24] M. A. Krasnoselskii, P. P. Zabreiko, E. I. Pustylnik and P. E. Sobolevskii, Integral Operators in Spaces of Summable Functions. Noordhoff Intnl. Publ.: Leiden, 1976.

[25] M. Krein and S. Krein, *On an inner characterisation of the set of all continuous functions defined on a bicompact Hausdorff space*, C. R. Acad. Sci. URSS **27** (1940), 427–430.

[26] U. Krengel, *Über den Absolutbetrag stetiger linearer Operatoren und seine Anwendung auf ergodische Zerlegungen*, Math. Scand. **13** (1963), 151–187.

[27] U. Krengel, *Remark on the modulus of compact operators*, Bull. Amer. Math. Soc. **72** (1966), 132–133.

[28] W. A. J. Luxemburg & A. C. Zaanen, *Compactness of integral operators in Banach function spaces*, Math. Ann. **149** (1963), 422–447.

[29] P. Meyer-Nieberg, Banach Lattices. Springer-Verlag: Berlin, Heidelberg, New York, 1991.

[30] M. Mokhtar-Kharroubi, *Compactness properties for positive semigroups on Banach lattices and applications*, Houston J. Math. **17** (1991), 25–38.

[31] R. J. Nagel and U. Schlotterbeck, *Kompaktheit von integraloperatoren auf Banachverbänden*, Math. Ann. **202** (1973), 301–306.

[32] H. Nakano, *Über das System aller stetigen Funktionen auf einem topologischen Raum*, Proc. Imp. Acad. Tokyo **17** (1940–41), 308–310.

[33] B. de Pagter, *Irreducible compact operators*, Math. Z. **192** (1986), 149–153.

[34] L. D. Pitt, *A compactness condition for linear operators on function spaces*, J. Operator Theory **1** (1979), 49–54.

[35] I. M. Popovici and D. T. Vuza, *Factoring compact operators and approximable operators*, Z. Anal. Anwendungen **9** (1990), 221–233.

[36] I. M. Popovici and D. T. Vuza, *Factorization of convolution operators*, An. Univ. Craiova Mat. Fiz.-Chim. **16** (1988), 1–8.

[37] H. H. Schaefer, Banach Lattices and Positive Operators. Springer: Berlin & New York, 1974.

[38] H. U. Schwarz, Banach Lattices and Operators. Teubner Texte: Leipzig, 1984.

[39] M. Talagrand, *Some weakly compact operators between Banach lattices do not factor through reflexive Banach lattices*, Proc. Amer. Math. Soc. **96** (1986), 95–102.

[40] A. W. Wickstead, *Extremal structure of cones of operators*, Quart. J. Math. Oxford Ser. 2 **32** (1981), 239–253.

[41] A. W. Wickstead, *Converses for the Dodds-Fremlin and Kalton-Saab theorems* (submitted).

[42] A. W. Wickstead, *Dedekind completeness of some lattices of compact operators* (submitted).

[43] A. W. Wickstead, *Spaces of operators with the Riesz Separation Property* (submitted).

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HIGHER ORDER SYMMETRY OF GRAPHS*

Ronald Brown

Symmetry in analogues of set theory.

This article gives background to and results of work of my student John Shrimpton [18, 19, 20]. It advertises the joining of two themes: groups and symmetry; and categorical methods and analogues of set theory.

Groups are expected to be associated with symmetry. Klein's famous *Erlanger Programm* asserted that the study of a geometry was the study of the group of automorphisms of that geometry.

The structure of group alone may not give all the expression one needs of the intuitive idea of symmetry. One often needs structured groups (for example topological, Lie, algebraic, order,...). Here we consider groups with the additional structure of a *directed graph*, which we abbreviate to *graph*. This type of structure appears in [13] and [17].

We shall associate with a graph A a group $\mathrm{AUT}(A)$ which is also a graph. The vertices of $\mathrm{AUT}(A)$ are the automorphisms of the graph A and the edges between automorphisms give an expression of "adjacency" of automorphisms. The vertices of this graph form a group, and so also do the edges. The automorphisms of A adjacent to the identity will be called the *inner automorphisms* of the graph A. One aspect of the problem is to describe these inner automorphisms in terms of the internal structure of the graph A.

* This paper is an account of a lecture "Groups which are graphs (and vice versa!)" given to the Fifth September Meeting of the Irish Mathematical Society at Waterford Regional Technical College, 1992.

The second theme is that of regarding the usual category of sets and mappings as but one environment for doing mathematics, and one which may be replaced by others. We use the word "environment" here rather than "foundation", because the former word implies a more relativistic approach.

The other environment we choose here is the category of directed graphs and their morphisms. We define this category, and then use methods analogous to those of set theory within this category. This allows set-theoretic intuition to be used to generate new results and methods, and is possible because of the "good" properties of this category of graphs. The background here is that of topos theory, which has given methods for considering many other environments for mathematics, and for comparing these environments.

Topos theory takes a relative rather than absolute viewpoint towards sets. The topos of sets is obviously an important, standard and basic kind of topos, but suffers from the defect of being somewhat boring, reflecting the fact that the objects of the topos, namely the abstract sets, are devoid of structure. The topos theory approach allows not only other versions of the category, or topos, of sets, but also allows comparison of different versions, through the notion of functor and natural transformation.

Thus different notions of set, or graph, can be evaluated by comparing the properties of the associated category. This global viewpoint has proved fruitful. One point of appearance was in topology, where the standard category of topological spaces was found not to have a function space with convenient properties. So different categories of topological spaces were proposed with "better" or more convenient function spaces.

The idea of emphasizing the categorical aspects of sets is not so familiar outside of category theory. For example, the article [1] does not mention any categorical approach. The traditional viewpoint is that sets are defined by the membership relation. There is, however, a strong argument that this approach is counter intuitive, since for many sets we wish to use, such as that of real numbers, it is very difficult to get one's hands on any but a small fraction of their members.

The categorical approach is that sets are defined by the relations between them, namely the functions, and this view has been strengthened by the success of topos theory. The book [14] is a good introduction to topos theory for those with a foundation in category theory. For an article relating the history of topos theory to notions of the foundations of sets, see [15]. The author emphasizes that the notion of topos was defined by Grothendieck as a replacement for the notion of topological space. Thus it was intrinsic to the definition that many different topoi were to be considered.

In the work of Lawvere, categories of structures other than sets are regarded as having intuitive value equal to that of the category of sets. That is, the category of sets is not regarded as a foundation for mathematics. Some words have to be said on the advantages of categorical methods, whose objectives and methodology have failed to be realized by some. The book by Reid [16] even writes: "The study of category theory for its own sake (surely one of the most sterile of intellectual pursuits) also dates from this time; Grothendieck himself can't necessarily be blamed for this, since his own use of categories was very successful in solving problems".

This quotation has aspects which should be noted. One is that it derides some vaguely specified group of colleagues as essentially unprofessional. A second is its lack of adventure. Let me propose a game: "I can think of a more sterile intellectual pursuit than you". A third is that it is hardly sensible to think of "blaming" Grothendieck for developments in mathematics. A fourth is its avoidance of historical analysis and of supporting evidence. This should be contrasted with McLarty's article [15].

A fifth is the view that the aim of mathematics is the solution of problems, which, by implication, are already formulated. By contrast, a historical view shows that the value of mathematics for other subjects, and for its own ends, is that it has developed language for:

- the study of patterns and structures;
- the formulation of problems;
- the development of methods of calculation and deduction.

The solution of problems is often a byproduct of this wider process and these wider aims. In this process, the study of an area for its own sake is often a necessary developmental stage. Judgements on the sterility or otherwise of such a study can be a matter of timing, or of gossip and snobbery, and are not always based on comparison and scholarship.

Does our education of mathematicians train them in the development of faculties of value, judgement, and scholarship? I believe we need more in this respect, so as to give people a sound base and mode of criticism for discussion and debate on the development of ideas.

The origins of category theory help to explain its utility. It arose from attempting to explain the meaning of the word "natural" in mathematics, and with a strong impetus from the axiomatic approach to homology theories, developed by Eilenberg and Steenrod, [6]. The original paper on the subject by Eilenberg and Mac Lane, [5], has an interesting discussion of the word "natural" in terms of the map $V \to V^{**}$ of a vector space into its double dual. To define natural required a definition of functors, and to define functors required a definition of category. This itself reflected also the growing realization that whenever a structure has been defined, it is usually necessary to consider also the morphisms of that structure.

By now, the general notions of limit and colimit, whose formulation was possible with the use of categories, and the later notion of adjoint functor, must be regarded as basic tools in mathematics. For example, the fact that a functor which is a left adjoint preserves colimits, while a right adjoint preserves limits, is a useful computational tool in many aspects of algebra and even combinatorics. Graduate books will probably have to give initial sections on basic concepts of category theory, in the same way as they have given basic sections on set theory, algebra and topology.

Category theory has been found useful for

- a global approach: i.e. constructions are defined by universal properties, which give the relation of the constructed object to all other objects;
- formulating definitions and theorems;

- carrying out proofs;
- discovering and exploiting analogies between various fields of mathematics.

Grothendieck's work on the foundations of algebraic geometry led him to develop a vast range of new categorical concepts. It is significant that his first important work was in analysis, and he brought to algebraic geometry a local-to-global approach. In algebraic geometry, it seems that "local" means "at a given prime p", and "global" means "over the integers". His approach was also to take concepts seriously, recognizing the effort required to "bring new concepts out of the dark" ([7]), and to spend a lot of effort in turning difficult results into a series of tautologies.

As one other recent example, and an indication of a wide literature, the paper by Joyal and Street, [8], illustrates how an algebraic development initially formulated for metamathematical reasons, and almost for its own sake, namely the notion of monoidal, or tensored, category, has found striking applications in concrete problems in knot theory, and string theory in physics.

One of the attractions of category theory is that the same algebraic tools are found applicable at several levels, and in a variety of areas. This feature is also found in groupoid theory, of which a survey was given by me in [3]. This notion has allowed the formulation of important extensions of group theory and of notions of symmetry.

Thus category theory is *par excellence* the method which enables the recognition and exploitation of many forms of analogy and comparison of structures. The point is that the algebraic study of the structure of a theory involves studying the categories and functors associated to the theory, and such a study leads to new algebraic notions of interest in their own right.

Applications to graph theory.

There are several unfamiliar aspects of this approach as applied to graph theory.

1) In this approach, it is essential to use a category of graphs and their morphisms. By contrast, it is not so easy to find a book on graph theory which defines a morphism of graphs.

2) An important categorical method used is that of *universal property*. In our setting, this defines a construction on graphs by the relation of the construction to all graphs. This may seem curious and far from logical. In fact, a construction by universal properties is analogous to a program, which when given an input of particular graphs, or graphs and morphisms between them, gives an output, namely new graphs and new morphisms. This analogy to programming is one reason why computer scientists have found the methods of category theory useful.

3) We lift to the category of graphs standard methods available in the category of sets and functions.

There are many possible definitions of graph and morphism of graph. We take one which gives for our purposes the "best" properties of the corresponding category. This again is an example of a "global" approach, and is simply a step or so up from a common approach in mathematics of considering for example all numbers, or all the symmetries of a square.

We deal here only with directed graphs. So for us a *graph* will mean a set A_E of edges, a set A_V of vertices and three functions $s, t : A_E \to A_V$, $\epsilon : A_V \to A_E$ such that $s\epsilon = 1$, $t\epsilon = 1$. Here s and t are the *source* and *target* maps. If $x, y \in A_V$, then $A(x, y)$ denotes the set of edges with source x and target y. Such an edge a is also written $a : x \to y$. A *loop* is an edge with the same source and target.

This defines in essence a directed graph in which each vertex v has an *associated loop* ϵv at that vertex. This extra structure makes no difference to the combinatorics of an individual graph, but makes a considerable difference to the allowable graph morphisms. The associated loop at a vertex v is often written • and given the vertex label v. Thus one of the simplest graphs, denoted I, is pictured as

$$0\bullet \longrightarrow \bullet 1.$$

A *morphism* of graphs $f : A \to B$ is a pair of functions $f_E : A_E \to B_E$, and $f_V : A_V \to B_V$ preserving the source and target maps, and ϵ. The implication is that f maps edges to edges, vertices to vertices, and f can map a general edge to the loop associated to a vertex. In effect, this means edges may be mapped to vertices.

The category $\mathcal{DG}$ of directed graphs has objects the graphs and arrows the morphisms, and $\mathcal{DG}(A,B)$ denotes the set of graph morphisms $A \to B$. Lawvere in [12] calls this the category of *reflexive* graphs.

This category has a *terminal object*, written $\bullet$, with the property that, for any graph A, the vertices of A are naturally bijective with $\mathcal{DG}(\bullet, A)$. The edges of any graph A are naturally bijective with $\mathcal{DG}(I, A)$.

Continuing with the categorical approach, we define the product of graphs.

A *product* of graphs A and B consists of a graph $A \times B$ with morphisms $p : A \times B \to A$, $q : A \times B \to B$ such that for any graph C the function

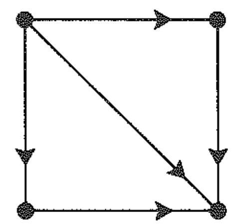
$$\mathcal{DG}(C, A \times B) \to \mathcal{DG}(C, A) \times \mathcal{DG}(C, B)$$

$$f \mapsto (pf, qf)$$

is a bijection. This says that a morphism to $A \times B$ is completely described by its component morphisms to A and B. The definition is also analogous to the law for numbers $(ab)^c = a^c b^c$.

It may be proved from the definition that the vertices of $A \times B$ are pairs of vertices from A and B, and the edges of $A \times B$ are pairs of edges from A and B. One way of proving this is to show that if $\mathcal{SETS}$ denotes the category of sets and functions, then the two functors $\mathcal{DG} \to \mathcal{SETS}$ given by the edges and the vertices have left adjoints, and so preserve limits, and in particular products. This deduction is one example of the "comparison" of environments referred to earlier. An important aspect of this procedure is that the product is defined by the universal property, which is the property that is most often used, and then a specific construction is deduced from the universal property. This verifies existence of the product.

As a typical example of the product of graphs, associated with the simplest graph I we have the product $I \times I$, illustrated by the following diagram:



Given sets B and C there is a set C^B of functions $B \to C$. In our category of graphs, the analogous construction is of course a *graph of morphisms* DIGRPH(B, C).

In the category of sets we have the standard *exponential law*

$$C^{A\times B} \cong (C^B)^A.$$

This corresponds to the law for numbers $c^{ab} = (c^a)^b$. In graph theory, we have the analogous law:

For graphs A, B and C, there is a natural bijection

$$\mathcal{DG}(A \times B, C) \cong \mathcal{DG}(A, \mathrm{DIGRPH}(B, C)).$$

Here the morphism graph DIGRPH(B, C) is in effect defined by this formula. From this formula, we can deduce the specific construction as follows.

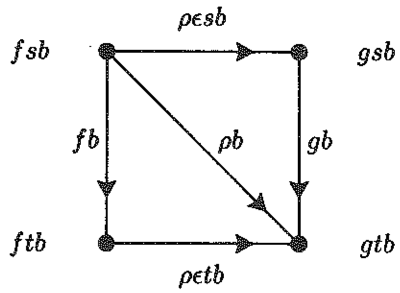
Let B and C be graphs. The graph DIGRPH(B, C) is to have vertices the morphisms of graphs $B \to C$ and to have edges the triples (ρ, f, g) such that f and g are morphisms of graphs $B \to C$ and $\rho : B \to C$ is a function from edges to edges such that if b is an edge of B then

$$s\rho b = fsb, \quad t\rho b = gtb.$$

Define

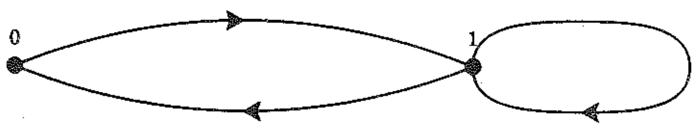
$$s(\rho, f, g) = f, \quad t(\rho, f, g) = g, \quad \epsilon(f) = (f, f, f).$$

Then each edge b of B yields the diagram



Comments.

1. If you define a directed graph by omitting ϵ, then product and morphism graph are defined, but the vertices of the morphism graph are not the morphisms of graphs. Instead, the morphisms correspond to the loops at vertices. From the categorical viewpoint, this is not surprising. The morphisms $B \to C$ should correspond to the morphisms $\bullet \to \mathrm{DIGRPH}(B, C)$, where $\bullet$ is the *terminal object* in the category, i.e. the graph such that there is exactly one morphism $A \to \bullet$ for any graph A. If the associated loop is omitted from the definition of graph, then the terminal object again has one vertex and one loop, and the morphisms of graphs are then not the vertices of the morphism graph, but are instead the loops of this morphism graph. The relations between these two categories of directed graphs are considered by Lawvere in [12].
2. There is another analogy between the category $\mathcal{DG}$ and the category of sets and functions. We can define in $\mathcal{DG}$ a graph Ω



and a morphism of graphs

$$\bullet \xrightarrow{\text{true}} \Omega,$$

called the *sub-object classifier* because it classifies subgraphs in a manner analogous to the way the inclusion

$$\{1\} \to \{0, 1\}$$

in sets classifies subsets via the characteristic function of a subset.

With this sub-object classifier, with the constructions defined earlier, and with the construction of limits (a more general notion than product), $\mathcal{DG}$ becomes what is called a *topos.* The name is due to Grothendieck, and was envisaged by him as a replacement of the notion of topological space by the category of sheaves on that space.

For our purposes, the idea is to carry out arguments in the topos $\mathcal{DG}$ as if it were the category of sets and functions, but never to use the law of the excluded middle. The reason for this is that the lattice of subgraphs of a given graph is not Boolean, since for example the complement $A \setminus (A \setminus B)$ of the complement $A \setminus B$ of a subgraph B is usually not the original subgraph B. Thus this theory is intuitionistic, an approach which is seen in this context as a practical mathematical tool for dealing with situations where the notion of membership is not the primary aspect. In the case of graphs, the "elements" have to be the vertices, but these capture only a small part of the structure. For more information on this approach in graph theory, see [12], while for the general body of theory, see the book by Mac Lane and Moerdijk, [14].

The exponential law in $\mathcal{DG}$ has a number of consequences. One is that there is a composition morphism

$$\mathrm{DIGRPH}(B, C) \times \mathrm{DIGRPH}(A, B) \to \mathrm{DIGRPH}(A, C)$$

which is associative and with identity. Hence

$$\mathrm{END}(A) = \mathrm{DIGRPH}(A, A)$$

has the structure of both a monoid and a graph. In the category of sets, monoids have maximal subgroups. This is also true in a topos. In the case of graphs, the maximal subgroup of the monoid $\mathrm{END}(A)$ is called

$$\mathrm{AUT}(A).$$

It is a group which is also a graph, or a graph which is also a group. Its set of vertices is the group Aut(A) of automorphisms of A.

Example: Let Δ_n be the complete graph on n vertices, and let D_n be the discrete graph on n vertices. These graphs have the same automorphism group, S_n, the symmetric group on n letters. But AUT(Δ_n) is the complete graph, while AUT(D_n) is discrete.

In the graph AUT(A), the automorphisms adjacent to the identity form a normal subgroup of Aut(A): these automorphisms are called the *inner automorphisms* of A.

This raises the problem of describing the inner automorphisms of a graph in terms of internal properties of the graph. The solution is given by Shrimpton, [19, 20], in terms of the notion of inner subgraph.

A subgraph B of a graph A is *inner* if it is maximal with respect to the following properties:

1. complete (i.e. the sets $B(x,y)$ have the same cardinality for all $x, y \in B_V$);
2. full (i.e. $B(x,y) = A(x,y)$ for all $x, y \in B_V$);
3. any automorphism of B extends to an automorphism of A which is the identity on the complement $A \setminus B$ of B in A.

Claim [19, 20]. *Any vertex belongs to a unique inner subgraph.*

Theorem [19, 20]. *An automorphism of a graph is inner if and only if it restricts to an automorphism of each inner subgraph.*

This suggests that the inner subgraphs are a kind of *atom* of symmetry of the graph.

The consideration of group-graphs leads to another new notion, the *centre* of a graph.

A *group-graph* is defined by Ribenboim in [17] to consist of groups G_E and G_V and morphisms $s, t : G_E \to G_V$, $\epsilon : G_V \to G_E$ such that $s\epsilon = t\epsilon = 1$. This concept has occurred elsewhere, for example as a 1-truncated simplicial group [13], and as part of the structure of a group-groupoid, as in Brown and Spencer, [4], where this is called a $\mathcal{G}$-groupoid. Loday in [13] found it natural to consider the subgroup [ker s, ker t] and to say that the group-graph is a *cat*^{1}*-group* if this subgroup is trivial. If it is not trivial, we can form the quotient $\gamma G_E = G_E/[\ker s, \ker t]$ with the induced morphisms to G_V giving γG the structure of cat^1-group. We call

$$(\ker s) \cap (\ker t) \subseteq \gamma G_E$$

the *second homotopy group* of G and write it $\pi_2 G$.

In particular, if $G = \mathrm{AUT}(A)$, then $\pi_2(G)$ is called the *centre* $Z(A)$ of the graph A. The centre is always an abelian group, and in fact is a module over $\mathrm{Out}(A) = \mathrm{Aut}(A)/\mathrm{Inn}(A)$. The aim is to describe this centre, in the case that A is finite, in terms of the structure of A.

To this end, we introduce in the following proposition an equivalence relation on the edges of a graph.

Proposition [19, 20]. *If A is a graph, then there is an equivalence relation on the edges of A given by x is equivalent to y if and only if there are inner subgraphs I and J of A such that sx, sy lie in I and tx, ty lie in J.*

Theorem [19, 20]. *If A is finite, then the centre $Z(A)$ of A is a direct sum of copies of the cyclic group of order 2, the number of copies being the number of equivalence classes of edges of A which contain multiple edge sets.*

Conclusion.

We have now shown that the study of categorical aspects of graph theory can lead to new problems, questions, and insights, and that it gives an interesting example of the relative viewpoint on set theory as exemplified by topos theory. Further work that might be done is in the area of "actions" of group-graphs, as well as the investigation of higher dimensional versions of AUT(A), such as the notion of automorphisms of ordered simplicial complexes.

The category $\mathcal{DG}$ is an example of what is called a *presheaf category*, namely a functor category $\hat{\mathcal{C}} = (\mathcal{SETS})^{\mathcal{C}^{op}}$ for a small category $\mathcal{C}$. The specific constructions outlined above for directed graphs are special cases of the fact that any such presheaf category $\hat{\mathcal{C}}$ is a topos (see Mac Lane and Moerdijk, [14]). These and other topoi yield a range of other "environments" for mathematics, or for a particular study, while types of categories other than topoi may be more suitable for other aims.

The notion of an internal group object in a category or in a topos is quite old. Thus the surprise is that the detailed study of this particular example, and the elucidation of the properties of the automorphism group-graph, had not been considered earlier. This suggests that there may be considerable mileage to be had from applying in new ways and in new places these and other concepts and methods of category theory.

Acknowledgements.

This paper gives my interpretation of what has been said by those who have taught me the categorical approach, particularly Tim Porter at Bangor, and many others at the Peripatetic Seminars in Sheaves and Logic, and at Category Theory meetings, and whom I would like to thank.

References

[1] J. Barwise and L. Moss, *Hypersets*, Math. Intelligencer **13**(no. 4) (1991), 31–41.

[2] M. Al-Ali, R. Brown, I. Morris, J. Shrimpton and C. D. Wensley, *Graphs of morphisms of graphs* (in preparation).

[3] R. Brown, *From groups to groupoids: a brief survey*, Bull. London Math. Soc. **19** (1987), 113–134.

[4] R. Brown and C. B. Spencer, *$\mathcal{G}$-groupoids, crossed modules, and the fundamental groupoid of a topological group*, Proc. Kon. Nederl. Akad. Wetensch. **79** (1976), 296–302.

[5] S. Eilenberg and S. Mac Lane, *The general theory of natural equivalences*, Trans. Amer. Math. Soc. **58** (1945), 231–294. Reprinted in S. Eilenberg and S. Mac Lane (eds.), Eilenberg-Mac Lane Collected Works, Academic Press (1986).

[6] S. Eilenberg and N. Steenrod, Foundations of algebraic topology. Princeton University Press, 1952.

[7] A. Grothendieck, *Private communication* (1987).

[8] A. Joyal and R. Street, *The geometry of tensor calculus*, Adv. Math. **88** (1991), 55–112.

[9] F. W. Lawvere, *An elementary theory of the category of sets*, Proc. Nat. Acad. Sci. U.S.A. **52** (1964), 1506–1511.

[10] F. W. Lawvere, *Metric spaces, generalized logic and closed categories*, Rend. Sem. Mat. Fis. Milano **43** (1973), 135–166.

[11] F. W. Lawvere, *Taking categories seriously*, Revista Colombiana Mat. **20** (1986), 147–178.

[12] F. W. Lawvere, *Qualitative distinctions between some toposes of generalized graphs*, Contemp. Math. **92** (1989), 261–299.

[13] J.-L. Loday, *Spaces with finitely many non-trivial homotopy groups*, J. Pure Appl. Algebra **24** (1982), 179–202.

[14] S. MacLane and I. Moerdijk, Sheaves in geometry and logic: a first introduction to topos theory. Springer, 1992.

[15] C. McLarty, *The uses and abuses of the history of topos theory*, Brit. J. Philos. Sci. **41** (1990), 351–375.

[16] M. Reid, Undergraduate algebraic geometry. Cambridge University Press, 1988.

[17] P. Ribenboim, *Algebraic structures on graphs*, Algebra Universalis **16** (1983), 105–123.

[18] J. Shrimpton, Cartesian closed categories of directed graphs, (University of Wales M.Sc. Thesis). U.C.N.W. (Maths. Preprint 88.5, 53 pp.), 1988.

[19] J. Shrimpton, Categorical aspects of symmetry, (University of Wales PhD Thesis), 1990.

[20] J. Shrimpton, *Some groups related to the symmetry of a directed graph*, J. Pure Appl. Algebra **72** (1991), 303–318.

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Abstract of Doctoral Thesis

TOTAL NEGATION IN GENERAL TOPOLOGY AND IN ORDERED TOPOLOGICAL SPACES

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This thesis was prepared in the Pure Mathematics Department of the Queen's University of Belfast under the supervision of Dr T. B. M. McMaster and was submitted to the Faculty of Science in April 1991. The external examiner was Professor Gary Faulkner of North Carolina State University. The degree of Ph.D. was awarded in July 1991.

Total negation is a procedure, formulated by Paul Bankston [Illinois J. Math. **23** (1979), 241–252], whereby each topological invariant P gives rise to another denoted anti-P, a space being called anti-P when none of its subspaces is P except for those whose point-sets, by virtue of their cardinalities, cannot sustain a non-P topology. This thesis begins by surveying the previous literature on the topic, and consolidates it by investigating the total negations of several invariants (including separability, first and second countability and separation axioms weaker than T_1) and by establishing previously unnoticed implications amongst a number of conditions related to anti-compactness.

Then the theory is extended in three principal directions. One of these arises from Brian Scott's characterization of those invariants P for which an invariant Q can be found whose total negation is P; here the circumstances in which Q may be taken to be hereditary are identified, and several results are obtained about the class of all such hereditary properties Q for a given P, for example, concerning whether or not this class possesses a weakest or a strongest member.

Another is an exhaustive investigation of the possibilities occurring when the anti-operation is iterated, especially with

regard to the repetitive patterns of invariants thus generated; a sample conclusion is that for any P either every topological space is anti^3-P or else anti^2-P =anti^3-P [J. Inst. Math. & Comp. Sci. (1990), 31–35].

The third extension is a broadening of these ideas into other categorical settings, beginning with an account of total negation in the context of ordered topological spaces in which the rôle played by *cardinality* of subspaces in the definition of anti-P is replaced by *order-type*. Many of the classical notions carry over to this setting, including versions of the iteration theorem and of the Scott characterization. The final chapter provides a categorical perspective for this material, indicating the possibility of parallel theories in groups, dimension theory, pure partial order etc. [Boll. Unione Mat. Ital., to appear].

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Research Announcement

A NON-CONFORMING FINITE ELEMENT METHOD FOR A SINGULARLY PERTURBED BOUNDARY VALUE PROBLEM

D. Adam, A. Felgenhauer, H.-G. Roos & M. Stynes

We analyse a new non-conforming Petrov-Galerkin finite element method for solving linear singularly perturbed two-point boundary value problems without turning points. The method is shown to be convergent, uniformly in the perturbation parameter, of order $h^{1/2}$ in a norm slightly stronger than the energy norm. Our proof uses a new abstract convergence theorem for Petrov-Galerkin finite element methods. Full details appear in [1].

Reference

[1] D. Adam, A. Felgenhauer, H.-G. Roos and M. Stynes, *A nonconforming finite element method for a singularly perturbed boundary value problem* (1992) (Preprint 1992–10, Mathematics Department, University College Cork).

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BENEFITS AND ADVANTAGES OF AN INTEGRATED MATHEMATICS AND COMPUTER SCIENCE DEGREE

T. C. Hurley

Introduction.

Computer science grew out of mathematics - it is surely enough to mention the names and contributions of Babbage, Boole, Hilbert, Von Neumann, Turing. On the other hand, it is clear that mathematics grew out of computation. The two areas are intimately related. Strenuous efforts have been made ever since computers were invented to separate mathematics and computer science. Modern techniques in both areas serve only as a reminder of how much each can be dependent on the other.

Difficulties.

In schools, mathematics is thought of as a 'closed' subject, whereas computer science is thought of as a 'technical' subject. Thus many creative individuals are turned off mathematics and computer science at an early stage.

A pure computer science degree is now considered in some quarters to be a *purely technical* training and not a proper education for a scientist or an engineer. One major employer is quoted as saying: "I only hire mathematicians and engineers–computer science graduates do not know how to *solve problems*".

Engineering and physical science degree programmes insist on a reasonable mathematical background. This is not true for computer science programmes and the graduate here needs this type of background even more than an engineer–consider, as an example, the situation where many computer science graduates are thrown into work which involves *Field Theory*.

'Programming' has been replaced by 'software engineering', with some emphasis on the 'engineering' aspect. Software engineers need a broad mathematical education. Computer science graduates have no mathematical equipment in which to analyse what they are trying to do.

Hoare, a well-known computer scientist, in his inaugural address made the following assertions:

- computer programs are mathematical expressions;
- a programming language is a mathematical theory;
- programming is a mathematical activity.

Computer science is dominated by the 'life-cycle' which obscures the mathematical dependency. It should be generating new formalism, modelling with formalism, constructing proofs and algorithms. It is stated that software engineers do not construct theories but apply *methods*, thus arguing that it is a *closed* problem and that computer systems now carry out mechanical methods. However the methods come out of a theoretical understanding. Everything in software engineering is seen in terms of the final product and the theoretical context is overlooked.

There are great difficulties in attracting students to mathematics degrees, especially from those who come into Arts or Science degree programmes with no specific subject in mind ('undenominated' programmes). We have to compete with other subjects which are *perceived* to be more vocational, are also perceived as *easier options* and do not have the high fallout rate that honours mathematics has.

Students of mathematics lack *motivation* and have *inadequate preparation*; this contributes to their subsequent subject choices.

On the other hand it is recognized that mathematicians, when they do solve problems, are unable to *communicate* their solution to others.

An integrated mathematics and computer science degree will go some way to rectifying many of these problems.

Common interests?

What areas ought mathematicians be interested in? In the software development process, *validation* and *verification* are now extremely important. "Get the *right system* and get the *system right*". The ideas of *modelling* and *proof* are emphasized again and again.

To quote a person working in the industry: "In the software engineering process, the use of mathematical ideas requires more resources at the initial stages but the **total** resources are less and an *enhanced* product ensues".

To be more specific, the following mathematical ideas should play an important part in the education of a software engineer:

- formal methods;
- modern logics;
- parallel processing;
- modelling, forecasting;
- complexity;
- computability;
- encryption and coding;
- statistics and probability (*networking* is a probabilistic entity);
- neural networks.

Benefits and advantages.

It is nice to make a list, so here goes! (Thanks to many articles and talks from which these have been taken.)

Benefits of an integrated degree (not necessarily in order):

- volatile and exciting area;
- focused programme;
- it still embodies *essence* of mathematics when carefully thought out;
- it facilitates development of PSQs (personal skills and qualities) through group and individual projects;
- vocational;
- accessibility to potentially good students;
- it prepares potential school teachers in both computer science and in mathematics;

- it is a laboratory based programme and can lead to better funding;
- modern applicable area;
- employment prospects for students are greatly increased;
- accessibility to funding agencies and politicians;
- it leads to active applications and involvement by the student who is encouraged to build his/her own system with great satisfaction;
- the emphasis is on *abstraction* and *proof.*
- it is not the 'death of proof' but a deeper understanding of what 'proof' is will ensue from experiment;
- students can see harder continuous mathematics in a new light;
- there is less emphasis on graduates as technicians and more emphasis on science;
- the image of mathematics and of computer science is enormously improved.

Employers' perspective.

Where is mathematics needed in industry? Do employers recognize that mathematics is needed? Again we might ask: "Do employers and colleagues within industry support mathematical activity?". These are difficult questions and the answers are of course dependent on the industry in question. The level of mathematical activity in different industries varies enormously. If we look at the worldwide context then we see that the industries using computer science may be roughly bracketed as follows:

1. Engineering houses, computer manufacturers, research laboratories, scientific users (such as the Meteorological Office).
2. Software houses.
3. Commercial users. These could be subclassified as:
   (a) utilities;
   (b) manufacturers;
   (c) finance institutions;
   (d) retail outlets.

These are listed in order of acceptability of the mathematical tradition. The ones within group 3 in fact have traditionally employed many non-graduates in their computing areas.

Those concerned with rigorous development also include the military and data-security teams within commerce and finance.

These classifications are on an international basis and Ireland has its own particular problems as a small open economy, with manufacturing but little research and development. Our employers also have the narrow view that graduates should be trained for a *specific job* but overlook how things are changing rapidly and that a well-trained mind with the ability to think and solve problems as and when they arise is what is needed.

Where do the graduates go?

There are now figures available from different countries as to where mathematics graduates get employment. The figures for Ireland cannot compare as we have so few honours mathematics graduates in comparison to other countries - e.g. the UK has the order of 4000 mathematics graduates each year, which, pro rata, would amount to about 350 graduates for Ireland. We are nowhere near this number.

The British *Sunday Times* on 8/8/91 reports: "Britain faces an acute shortage of mathematicians". If this is so, where does this leave us? Many of our good mathematics students go into engineering and the school culture is such that those good at mathematics are encouraged to apply there. Would those good at English be encouraged to do a degree in, say, accountancy?

Mathematics is not looked on as a career in itself and nowadays additionally some of the best mathematicians are attracted into business and commerce. If they are good at mathematics and must pursue a career in business, why is it not pointed out that a degree in mathematics and economics is a much better preparation? [1]

[1] See the article by Joel Franklin on *Mathematical methods of economics* in the American Math. Monthly, **90** (1983), 229-244. Among other things in this article it is pointed out that seven of

In the UK, 26% of mathematics graduates[2] go into the computer industry and 52% into finance. Of the mathematics graduates in Ireland that did *not* go on to further study, 82% (1991 figures) went into financial work and computing.[3] Graduates with an *integrated* degree would be much better prepared for careers in these areas.

Information Technology.

One of the buzz-words at the moment is information technology (IT). I, and many others, doubtless, are confused at what exactly IT is. It can mean different things to different persons, depending on whether you are a scientist, engineer, business person, sociologist, industrialist, philosopher or psychologist. The EU has classified IT under five headings. I am grateful to Pat Fitzpatrick for this information.

1. Software engineering or knowledge-based information systems.
2. AI (artificial intelligence).
3. VLSI (very large scale integration).
4. Communications.
5. Human interface.

Of these, only one, the human interface does not require a substantial mathematical background–perhaps. Where does mathematics come into these areas? Pat Fitzpatrick again has some of the answers. *Algorithm design* is fundamental for VLSI, *logic design, LISP* for AI and *coding theory, cryptography, digital signalling* are all areas of importance in communications. The importance of mathematical ideas to software engineering has already been dealt with. Even within the human interface, the

the previous twelve Nobel prizes in Economics 'involved work that is heavily mathematical'.

[2] Interestingly, the career booklet for mathematical graduates contains computer science as a subsection.

[3] Statistics on graduate careers only give destination for the year after first graduation.

ideas behind *security* and *authentication* are now increasingly important.

Computer Algebra[4] systems.

It is my opinion that it is inconceivable that a graduate in mathematics, and I would argue now a graduate in computer science, would nowadays not have had experience of computer algebra systems. It is like saying that a chemistry graduate never did any laboratory work! A graduate in many years time will continue to use mathematics if he/she realizes that some of the hard calculations and theoretical background, which he/she has probably forgotten, can be done by machine.

Computer algebra systems are likely to become as useful to scientists and engineers as word-processors and data-bases have become to all.

Cohen remarks: "A mediocre mathematician with a computer might be able to simulate the creative powers of a top notch mathematician with pen and paper". How much more could a top-notch mathematician produce?

Packages like MAPLE, MATHEMATICA, REDUCE, CAYLEY (to be replaced by MAGMA soon), GAP, AXIOM should be, in fact **must be**, integrated within our courses. From our experience it is much easier to integrate these within an integrated mathematics and computing degree, where an element of laboratory and experimental work already exists.

Generally speaking a package like MAPLE, REDUCE or MATHEMATICA would go well with analysis-type courses or applied/mathematical physics courses, and one of CAYLEY (soon to be MAGMA), GAP or maybe AXIOM[5] should be integrated within abstract algebra courses such as group theory, field theory, coding theory or even homological algebra. Also MATRIX, for

[4] Computer Algebra is also often referred to as *symbolic manipulation.*

[5] AXIOM is a package which has great potential, especially when you need to build your own abstract system, but is not available for many machines yet, is expensive and difficult to use.

linear algebra and linear programming, is an excellent CAL (computer assisted learning) package which goes down well with students. In addition, MACTUTOR, for MACs, is an excellent all-purpose CAL package.

Use of computer algebra does however tend to increase the workload on the instructor and many mathematics departments and colleges are as yet unwilling to recognize this fact and indeed do not recognize the importance of considering mathematics as a laboratory-based subject.

Leibnitz stated: ".... it is unworthy of excellent men to lose many hours like slaves in the labour of calculation, which could safely be relegated to someone else if the machine were used".

Syllabus.

There are various suggestions as to what should be included in a syllabus for a joint degree. Not all areas can be covered and choices must be made. What is important is that a good *scientific* training in *both* mathematics and computer science should be an essential part of any programme. Some might argue that a language and/or business skills and/or placement should be part of the programme but incorporating two major subjects does not leave much time for anything else, even within a four-year course. Projects in all years will develop PSQs.

There can be **core** courses and options to suit individual tastes. It is important to note however that this an *integrated* programme and that although some of the mathematics and computer science courses are independent, the programme should be drawn up by reference to common areas of interest and dependency. It is not a matter of simply combining the subjects as in a traditional two-subject degrees.

To spell out a full syllabus would be pointless here but some suggestions on related areas that could be included are given below–fundamental courses are in *addition* to these.

Within mathematics, the importance of *discrete mathematics* to computer science is fundamental and has now been recognized as such but this is only part of the picture. Discrete mathematics should include algorithms, recursive function theory, Boolean

algebra, logic and circuit design. Analysis courses are also an important element and must be compulsory in the early years. A good idea which works well at UCG is to have courses on metric spaces *and* fractal geometry. Other ideas to think about: have a course in number theory *and* cryptography; include coding theory with field theory, and semigroups and machines with a group theory course. Category theory: "Categories themselves are the models of an essentially algebraic theory and nearly all derived concepts are finitary and algorithmic in nature" (John Gray in *Computational Category Theory*). This is all good mathematics.

Numerical analysis cuts across both areas and could be included either within the mathematics core or within the computer science core areas.

In computer science, courses on programming, operating systems, networking and communications, data bases, architecture, modelling, algorithms, computability, complexity, graphics, parallel processing, artificial intelligence and logical aspects of computing, would appear to be fundamental. Consideration should also be given to courses on automated reasoning and neural networks which would fit in well with the mathematics. In general, the computer science element in a joint programme should be directed more to *software* considerations.

Everyone has their own favourite language but at the risk of upsetting some persons I would suggest that C and C++ are the most useful (*and* mathematically oriented) languages now used in science and industry. This also fits in well with operating systems as many of these were originally written in C.

Conclusion.

The programme suggested is what is needed and will train *both* mathematicians and computer scientists for worthwhile careers. It will also satisfy the needs of industry and the commercial world.

Both subjects have much to learn from one another. The debate will continue!

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THE IRISH INTERVARSITY COMPETITION IN MATHEMATICS

Timothy Murphy

What, in your opinion, is the next term in the sequence

$$3, 5, 1, 15, 11, 10?$$

This will be a cinch. Easy one to start with. Thanks, Des. Knew he was a good sort.

Let's see. Probably squares minus 1 ... that will explain the 15, anyway. No, doesn't seem to be that. Maybe it's not quite as simple as I thought ...

The Irish Intervarsity in Mathematics came out of the fertile brain of Des MacHale. It was a natural extension of the Superbrain Competition that Des has been running in UCC since 1984. (The question above was the opening problem in the 1987 Superbrain paper.)

I've got it! Should have thought of that earlier. It's obvious. Just a common-or-garden code, 1 for A, 2 for B, and so on. Let's see... CEAOKJ.... What language is this? Maybe the Viking name for Cork? Well, it was worth a try, anyway.

The Intervarsity was first held in Cork, in 1990. It moved to TCD for the next 2 years; and UCG hosted the event this year.

The competition was won by TCD in 1990 and 1991, and by UCC in 1992 and 1993. (UCC and UCD tied in 1993; but the prize went to UCC for the best individual result.)

Although the competition is mainly a team event, there is also an individual winner each year. Paul Harrington of TCD won in 1990; Aiden O'Reilly of Maynooth in 1991; Cian Dorr of UCC in 1992; and Peter Hegarty of UCC in 1993.

Maybe it's the number of steps on the stairs in the UCC Maths Dept ... Cork bus numbers ... The ages of Des MacHale's children, in alphabetical order ... The number of moons of the planets ... The winners of the Eurovision Song Contest ...

I've always suspected that man MacHale had a sadistic streak. Now calm down. I've only taken 20 minutes on this question so far. If he's a sadist, we must all be mathematical masochists to sit here and take this sort of thing from him. Everyone else seems to be scribbling away. Even that awful ass from UCD. What was it Tartakower said, "Why am I always being beaten by fools?"

Come on, pull yourself together. Only 25 minutes gone. Let's abandon this question. But now I've invested so much time in it. I know it must really be simple.

The universities in Northern Ireland are invited each year, and were expected in 1991, but didn't materialize. It would be nice to make a special effort to persuade them to take part next year. Perhaps with a promise of a meeting in Queens' the following year?

Women are also conspicuous by their absence. I think there were 2 in Maynooth's team last year, and that was about it. I wonder why? In TCD the students organize a selection test (which Richard Timoney and I usually set) but very few women will take part, even though they constitute some 30% of our student numbers (in maths). In 1991 Helen Joyce—who was our best student for some years as far as exam results are concerned (she went on to get a distinction in Cambridge Part III)—absolutely refused, even under extreme pressure.

I've got it! Not the letters themselves, but the numbers of letters in the words. A well-known saying, with 5 letters in the first word, 3 in the second, and so on. Well, what has 15 letters? Intervarsity? Not quite. Differentiation? Maybe.

I wonder if Kraft-Ebbing had a category for people like MacHale. Does he belong to a recognizable criminal type? Don't you see the similarity between his features and those of Hannibal Lector?

The questions in the Intervarsity were set by Des MacHale in

the inaugural year 1990, and again this year. Richard Timoney and I set the paper in the 2 intervening years.

What sort of questions do we set? Well, without saying this on oath, they shouldn't require much if anything beyond Leaving Certificate standard. And though they shouldn't be too predictable, there are certain recognizable families of problems, or perhaps one should say, families of solutions.

First, there are the problems involving moduli. There was a nice one of these in the 1987 Superbrain:
Is 31415314155314156314157314158314159 a prime?

So that must be it! At last. Fancy taking all that time to hit on it. It's just a calculation modulo n, for some n. Probably just multiplication by a, for some a. So that's it ... just have to work out n and a. I suppose n must be 16, since the largest number is 15. And a is ... Damn, I was sure I had it.

Then there are the infinite series to be summed. Here there seem to be 2 common themes: firstly, differentiating power-series and substituting (usually $x = 1$); secondly, expressing the nth term $a(n)$ in the form

$$a(n) = b(n+1) - b(n),$$

where $b(n) \to 0$, so that $\sum_{1}^{\infty} a(n) = b(1)$. I rather liked a variant of this I hadn't met before, using the relation

$$\tan^{-1} a - \tan^{-1} b = \tan^{-1} \frac{a-b}{1+ab},$$

which is just another way of saying

$$\tan(\theta - \phi) = \frac{\tan\theta - \tan\phi}{1 + \tan\theta\tan\phi}.$$

A pretty example of this is the sum

$$\sum_{1}^{\infty} \tan^{-1} \frac{1}{2n^2}.$$

Another is

$$\sum_1^\infty \tan^{-1}\frac{1}{n^2+n+1}.$$

A series surprisingly susceptible to the same difference technique is:

$$\sum_1^\infty \frac{2^n}{2^{2^n}+1}.$$

What do you mean, ten minutes more? Oh my God, my watch has stopped. Wait a moment. Inspiration, where are you? I never liked that man MacHale. Did you notice his eyebrows? The crime rate has rocketed since he started his Superbrain. Surely parents don't expect their children to be subjected to this kind of thing when they send them to college.

Then there are the 'sporadic' questions—once-off, never seen before and never to be seen again. There was one like that in the 1990 Intervarsity: Find any solution in positive integers of

$$x^x y^y = z^z.$$

The School of Maths in TCD ground to a halt for a week, as we all looked for solutions; and I noticed the mathematicians from UCD looking very tired and emotional at that time.

What do you mean, is that all I've written? I've got better things to do than sit around all day answering silly questions. I'm going for a drink. You see what that man has done to me, I never drink at this time of day.

OK, it must have been easy. Shall I ask that nasty type sitting in front of me, who spent the entire time scribbling. No. It would be too shaming. There are some things it is better not to know.

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