

A NEW GEOMETRIC INEQUALITY

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Abstract: We prove the conjecture that a triangle whose three vertices lie in the three sides of a larger triangle must have perimeter at least as large as that of one of the other small triangles which are created by its inscription there; we also give proofs of some related results.

Introduction

On reading the abstract above, one might suspect that the title of this article ought to have been followed by a question mark. Tom Laffey, who first drew my attention to this conjecture, proved below as Theorem 3, pointed out that it had been listed as an unsolved problem by Kazarinoff [1, p78] and that, had it been proved in the meantime, it would most likely have appeared in the compendious work [2] — where it is not included.

It might be of interest to TeX enthusiasts if I add a little personal note here before embarking on the proof. Unlike most problems we encounter in modern mathematics, questions of this sort can be settled quickly and almost with certainty by using a computer. It is therefore worthwhile to try this avenue before expending time on possibly futile mathematical calculations. Believe it or not, my choice of language for testing the hypothesis was Knuth's character drawing programme METAFONT. It turns out that a METAFONT programme for this type of task is shorter and cleaner than one written in a standard all-purpose programming language, and that, provided one is careful to avoid arithmetic overflow, it is also accurate and quick. My short programme tested 800 inner triangles, chosen with partial randomness, in each of 1,000 outer triangles, also chosen with partial randomness. The programme took no more than a few minutes to write. It made the 800,000

tests and failed to find a counterexample to the conjecture. Given the nature of the conjecture, in particular the continuity involved in it, this made it at least as likely to be true as Fermat's Last Theorem.

Notation If A , B and C are points in a plane, we shall use $\sigma(ABC)$ to denote the sum of the three lengths $|BC|$, $|CA|$ and $|AB|$. If ABC is a triangle, then ΔABC will denote its area; otherwise ΔABC should be understood to be 0.

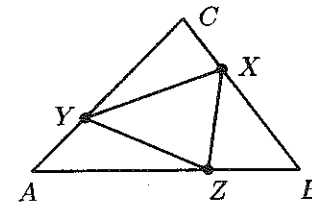
Perimeter theorems

It is not difficult to see that a proof of the truth of the conjecture will follow from a few technical manoeuvres, and we shall demonstrate that such is the case, if we can establish first a related result, which we present now as Theorem 1.

Theorem 1. Suppose ABC is a triangle and suppose X , Y and Z are points lying on the lines BC , CA and AB respectively. Then

$$\max\{\sigma(AYZ), \sigma(XBZ), \sigma(XYC)\} \geq \frac{1}{2}\sigma(ABC)$$

with equality if and only if X , Y and Z are the mid-points of the line segments $[BC]$, $[CA]$ and $[AB]$ respectively.



Proof: We denote by a , b and c the lengths of the line segments $[BC]$, $[CA]$ and $[AB]$ respectively. When X , Y and Z are the mid-points described, the equality is well known. We assume, therefore, without loss of generality, that $|AZ| > \frac{1}{2}c$.

Now if $|CY| \leq \frac{1}{2}b$ and we denote the mid-points of $[AC]$ and $[AB]$ by M and N respectively, then we have

$$\begin{aligned}\sigma(AYZ) &= |AM| + |MY| + |YZ| + |ZN| + |NA| \\ &> |AM| + |MN| + |NA| = \frac{1}{2}\sigma(ABC)\end{aligned}$$

and the required inequality follows. We may therefore assume that $|CY| > \frac{1}{2}b$. Similarly, we may assume that $|BX| > \frac{1}{2}a$. We now define r , s and t to be the strictly positive real numbers given by the equations

$$|BX| = \frac{1}{2}a + r, \quad |CY| = \frac{1}{2}b + s, \quad |AZ| = \frac{1}{2}c + t.$$

Now

$$\begin{aligned}\sigma(AYZ) &\leq \frac{1}{2}\sigma(ABC) \\ &\Rightarrow \frac{1}{2}c + t + \frac{1}{2}b - s + \\ &\quad \sqrt{\left(\frac{1}{2}c + t\right)^2 + \left(\frac{1}{2}b - s\right)^2 - 2\left(\frac{1}{2}c + t\right)\left(\frac{1}{2}b - s\right)\cos A} \\ &\leq \frac{1}{2}a + \frac{1}{2}b + \frac{1}{2}c \\ &\Rightarrow \left(\frac{1}{2}c + t\right)^2 + \left(\frac{1}{2}b - s\right)^2 - 2\left(\frac{1}{2}c + t\right)\left(\frac{1}{2}b - s\right)\cos A \\ &\leq \left(\frac{1}{2}a + s - t\right)^2 \\ &\Rightarrow \frac{1}{4}(b^2 + c^2 - 2bc\cos A) + ct - bs - (bt - cs - 2st)\cos A \\ &\leq \frac{1}{4}a^2 + sa - ta - 2st \\ &\Rightarrow (a + b + c)(t - s) \leq (bt - cs - 2st)(1 + \cos A) \\ &\Rightarrow 2bc(t - s) \leq (bt - cs - 2st)(b + c - a) \\ &\Rightarrow 0 \leq bt(b - c - a) + cs(b - c + a) - 2st(b + c - a) \\ &\Rightarrow 0 \leq \frac{b}{s}(b - c - a) + \frac{c}{t}(b - c + a) - 2(b + c - a).\end{aligned}$$

Similarly

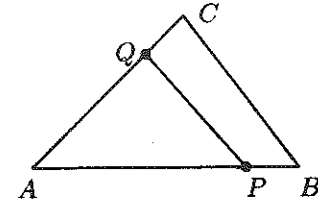
$$\sigma(XBZ) \leq \frac{1}{2}\sigma(ABC) \Rightarrow 0 \leq \frac{c}{t}(c - a - b) + \frac{a}{r}(c - a + b) - 2(c + a - b)$$

and

$$\sigma(XYC) \leq \frac{1}{2}\sigma(ABC) \Rightarrow 0 \leq \frac{a}{r}(a - b - c) + \frac{b}{s}(a - b + c) - 2(a + b - c).$$

That these three inequalities cannot be simultaneously satisfied is clear, because their addition would lead to the absurd $0 \leq -2(a + b + c)$. The theorem follows. □

Lemma 2. Suppose ABC is a triangle and P and Q are points on the sides AB and AC respectively, with $|PB| = |QC| > 0$. Then $|PQ| < |BC|$.



Proof: Set $|AB| = c$, $|CA| = b$, $|BC| = a$ and $|PQ| = d$.

Then

$$\begin{aligned}|BC|^2 - |PQ|^2 &= (c^2 + b^2 - 2bc\cos A) - \\ &\quad ((c - d)^2 + (b - d)^2 - 2(c - d)(b - d)\cos A) \\ &= -2d^2 + 2cd + 2bd + 2(d^2 - bd - cd)\cos A \\ &= 2d(b + c - d)(1 - \cos A) > 0,\end{aligned}$$

and the lemma is proved. □

Theorem 3. Suppose ABC is a triangle and X , Y and Z are points in the line segments $[BC]$, $[CA]$ and $[AB]$ respectively.

Then

$$\sigma(XYZ) \geq \min\{\sigma(AYZ), \sigma(XBZ), \sigma(XYC)\}$$

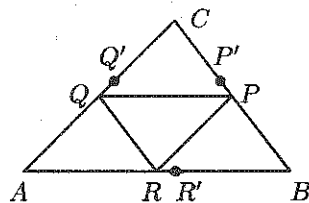
with equality if and only if X , Y and Z are the mid-points of the line segments $[BC]$, $[CA]$ and $[AB]$ respectively.

Proof: The function f defined on the compact set $[BC] \times [CA] \times [AB]$ by

$$f(X, Y, Z) = \sigma(XYZ) - \min\{\sigma(AYZ), \sigma(XBZ), \sigma(XYC)\}$$

is continuous and therefore attains its minimum value. Suppose this minimum value is attained at (P, Q, R) . Since f has the value 0 when X , Y and Z are the mid-points of the line segments $[BC]$, $[CA]$ and $[AB]$ respectively, we have $f(P, Q, R) \leq 0$. It is easily verified that if any of X , Y or Z coincides with any of A , B or C then $f(X, Y, Z) > 0$. It follows that P , Q and R are internal points of their respective line segments.

We want to establish firstly that the three quantities $\sigma(AQR)$, $\sigma(PBR)$ and $\sigma(PQC)$ are equal. To this end, we accept, without loss of generality, that $\sigma(AQR) \leq \sigma(PBR) \leq \sigma(PQC)$.



Suppose now that $\sigma(AQR) < \sigma(PBR)$ and consider an internal point R' of the line segment $[BR]$ which is close enough to R to ensure that $\sigma(AQR') < \sigma(PBR')$. Note that it is true in any case that $\sigma(PBR') < \sigma(PBR) \leq \sigma(PQC)$.

So we have

$$\begin{aligned} f(P, Q, R') &= \sigma(PQR') - \sigma(AQR') \\ &= \sigma(PQR) - \sigma(AQR) - (|PR| + |RR'| - |PR'|) \\ &< \sigma(PQR) - \sigma(AQR) \\ &= f(P, Q, R) \end{aligned}$$

contradicting the minimality of f at (P, Q, R) . We must therefore infer that $\sigma(AQR) = \sigma(PBR)$.

Suppose now that $\sigma(AQR) < \sigma(PQC)$ and consider internal points P' and Q' of the line segments $[PC]$ and $[QC]$ respectively, with $|PP'| = |QQ'|$ and this quantity being small enough to ensure that the inequalities $\sigma(AQ'R) < \sigma(P'Q'C)$ and $\sigma(P'BR) < \sigma(P'Q'C)$ hold. We note that Lemma 2 implies that $|PQ| > |P'Q'|$.

Then

$$\begin{aligned} \sigma(P'Q'R) - \sigma(P'BR) &= \sigma(PQR) - \sigma(PBR) + |Q'R| + |P'Q'| \\ &\quad - |QR| - |PQ| - |PP'| \\ &= \sigma(PQR) - \sigma(PBR) - (|PQ| - |P'Q'|) \\ &\quad - (|Q'Q| + |QR| - |Q'R|) \\ &< \sigma(PQR) - \sigma(PBR) \\ &= f(P, Q, R). \end{aligned}$$

Similarly

$$\begin{aligned} \sigma(P'Q'R) - \sigma(AQ'R) &< \sigma(PQR) - \sigma(AQR) \\ &= f(P, Q, R). \end{aligned}$$

It follows that $f(P', Q', R) < f(P, Q, R)$, so that minimality of f at (P, Q, R) is once again contradicted. We must therefore have $\sigma(PBR) = \sigma(AQR) = \sigma(PQC) = q$, say.

Now

$$\begin{aligned} q &\geq \sigma(PQR) = \sigma(AQR) + \sigma(PBR) + \sigma(PQC) - \sigma(ABC) \\ &= 3q - \sigma(ABC), \end{aligned}$$

so that $q \leq \frac{1}{2}\sigma(ABC)$. It follows from Theorem 1 that equality holds and that P , Q and R are the mid-points of the line segments $[BC]$, $[CA]$ and $[AB]$ respectively, and the theorem is proven. □

Area theorems

One might expect area analogues to the perimeter theorems of the last section, and one should be right to do so. Indeed, the analogue of Theorem 3 was known to Kazarinoff [1], though he simply states that it is true without giving a reference to a proof.

Theorem 4. Suppose ABC is a triangle and X , Y and Z are interior points of the line segments $[BC]$, $[CA]$ and $[AB]$ respectively. Then

$$\Delta XYZ \geq \min\{\Delta AYZ, \Delta XBZ, \Delta XYC\}$$

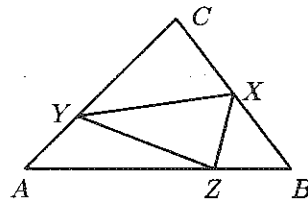
with equality if and only if X , Y and Z are the mid-points of the line segments $[BC]$, $[CA]$ and $[AB]$ respectively.

A proof of Theorem 4 can be effected rather easily by setting up a function which attains its bound and manipulating perturbations, only provided the analogue of Theorem 1 has first been established. That analogue, given below as Theorem 5, is much easier to demonstrate than Theorem 1. It is left to the reader to supply a proof of Theorem 4 using Theorem 5.

Theorem 5. Suppose ABC is a triangle and suppose X , Y and Z are points lying on the lines BC , CA and AB respectively. Then

$$\min\{\Delta AYZ, \Delta XBZ, \Delta XYC\} \leq \frac{1}{4}\Delta ABC$$

with equality if and only if X , Y and Z are the mid-points of the line segments $[BC]$, $[CA]$ and $[AB]$ respectively.



Proof: Evidently

$$\Delta AYZ \geq \frac{1}{4}\Delta ABC \Rightarrow 4|AY||AZ| \geq bc$$

$$\Delta XBZ \geq \frac{1}{4}\Delta ABC \Rightarrow 4|BZ||BX| \geq ca$$

and

$$\Delta XYC \geq \frac{1}{4}\Delta ABC \Rightarrow 4|CX||CY| \geq ab,$$

where a , b and c denote the lengths of the line segments $[BC]$, $[CA]$ and $[AB]$ respectively.

Multiplying the three inequalities at the right, we get

$$4|AZ||ZB| \times 4|AY||YC| \times 4|CX||XB| \geq a^2b^2c^2.$$

Since $|AZ| + |ZB| = c$, $|AY| + |YC| = b$ and $|CX| + |XB| = a$, it follows that the three inequalities at the left can be simultaneously satisfied only if X , Y and Z are the mid-points of their respective sides. □

References

- [1] N. D. Kazarinoff, *Geometric Inequalities*. Math. Ass. of America, 1961.
- [2] D. S. Mitrinović, J. E. Pečarić and V. Volenec, *Recent Advances in Geometric Inequalities*. Kluwer: Dordrecht-Boston-London, 1989.

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