IRISH MATHEMATICAL SOCIETY

cumann matamaitice na héireann

BULLETIN

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IRISH MATHEMATICAL SOCIETY
BULLETIN

The aim of the Bulletin is to inform Society members about the activities of the Society and about items of general mathematical interest. It appears twice each year, in March and December. The Bulletin is supplied free of charge to members; it is sent abroad by surface mail. Libraries may subscribe to the Bulletin for IR£20.00 per annum.

The Bulletin seeks articles of mathematical interest written in an expository style. All areas of mathematics are welcome, pure and applied, old and new. The Bulletin is typeset using TEX. Authors are invited to submit their articles in the form of TEX input files if possible, in order to ensure speedier processing.

Correspondence concerning the Bulletin should be addressed to:

Irish Mathematical Society Bulletin
Department of Mathematics
University College
Dublin
Ireland

Correspondence concerning the Problem Page should be sent directly to the Problem Page Editor at the following address:

Faculty of Mathematics
Open University
Milton Keynes, MK7 6AA
UK

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NOTES ON APPLYING FOR I.M.S. MEMBERSHIP

1. The Irish Mathematical Society has reciprocity agreements with the American Mathematical Society and the Irish Mathematics Teachers Association.

2. The current subscription fees are given below.

Institutional member	IR£50.00
Ordinary member	IR£10.00
Student member	IR£4.00
I.M.T.A. reciprocity member	IR£5.00

The subscription fees listed above should be paid in Irish pounds (puint) by means of a cheque drawn on a bank in the Irish Republic, a Eurocheque, or an international money-order.

3. The subscription fee for ordinary membership can also be paid in a currency other than Irish pounds using a cheque drawn on a foreign bank according to the following schedule:

If paid in United States currency then the subscription fee is US$18.00.
If paid in sterling then the subscription fee is £10.00 stg.
If paid in any other currency then the subscription fee is the amount in that currency equivalent to US$18.00.

The amounts given in the table above have been set for the current year to allow for bank charges and possible changes in exchange rates.

4. Any member with a bank account in the Irish Republic may pay his or her subscription by a bank standing order using the form supplied by the Society.

5. The subscription fee for reciprocity membership by members of the American Mathematical Society is US$10.00.

6. Subscriptions normally fall due on 1 February each year.

7. Cheques should be made payable to the Irish Mathematical Society. If a Eurocheque is used then the card number should be written on the back of the cheque.

8. Any application for membership must be presented to the Committee of the I.M.S. before it can be accepted. This Committee meets twice each year.

9. Please send the completed application form with one year's subscription fee to

The Treasurer, I.M.S.
Department of Physics
Regional Technical College
Cork
Ireland

IMS CONSTITUTION AND RULES

Accepted unanimously by the Society
20 December 1993

1. The Body Corporate hereby constituted shall be known in English as The Irish Mathematical Society and in Irish as Cumann Matamaitice na hÉireann and shall hereinafter be called the Society.

2. The Society is incorporated for the purpose of promoting and extending the knowledge of mathematics and its applications. Activities proper to the Society shall include the following:
(i) holding meetings of the members of the Society and visitors introduced by them,
(ii) publishing and distributing the Bulletin of the Society,
(iii) organizing and supporting conferences, lectures, and discussions on subjects of special and general interest to mathematicians,
(iv) discovering and making known the views of the members of the Society on mathematical matters of public interest,
(v) co-operating with other organizations to achieve the purpose of the Society.

3. The officers of the Society, hereinafter called the Officers, shall consist of a President, a Vice-President, a Treasurer, and a Secretary. Only persons who are Ordinary members of the Society may hold Office.

4. The governing body of the Society, hereinafter called the Committee, shall consist of the Officers and at least eight other Ordinary members of the Society. The Committee shall meet at least twice during each year, the President to be the convener. The quorum for each Committee meeting shall be five members of the Committee, including at least two Officers.

5. Membership of the Society shall be classified as follows:
   (i) Ordinary membership,
   (ii) Institutional membership,
   (iii) Honorary membership,
   (iv) Student membership.

6. Any person may apply to the Treasurer for Ordinary membership. Any institution may apply to the Treasurer for Ordinary or Institutional membership. The election of an Ordinary or Institutional Member shall rest with the Committee, whose decision shall be made at the first meeting of the Committee that follows the receipt by the Treasurer of the proper application for membership.

7. Any three Ordinary members of the Society may propose a candidate for election to Honorary membership by presenting such a proposal to the Committee. Deciding whether or not to accept such a proposal shall rest with the Committee. If the proposal is accepted by the Committee it shall then be voted upon at the next general meeting of the Society and carried if and only if there is no vote against it.

8. A candidate for election to Student membership must be a registered student at an institution that is an Institutional member of the Society and may be proposed for membership only by that institution. The election of a Student member shall rest with the Committee. Student membership shall be for one year only but may be renewed if a new proposal for membership is made. (In these Articles and Rules *year* always means the calendar year from the first day of January until the thirty first day of December.)

9. Every Ordinary and Institutional member shall pay a subscription fee to the the Society at the times and of the amounts specified in the Rules.

10. The Committee may agree with any other mathematical society that any member of that society who is also an Ordinary member of the Irish Mathematical Society may be designated a *reciprocity member* of the Society. The Committee is empowered to specify a special subscription fee for reciprocity members. Such an agreement with another mathematical society shall be made only if that society admits members of the Irish Mathematical Society as reciprocity members in like manner.

11. There shall be at least two general meetings of the Society in each year; the last general meeting held in a year shall be known as the *Annual General Meeting*. No motion may be passed at a meeting of the Society unless at least seven Ordinary members are present when it is proposed and when it is decided upon.

12. At each Annual General Meeting an election shall be held to fill each Office that would otherwise be vacant in the following year. A single election shall then be held to fill a sufficient number of seats on the Committee to ensure that there are twelve members on the Committee in the following year. Any contested election shall be decided by secret ballot using the single transferable vote.

13. The Committee shall have the power to co-opt one or more additional members at any time provided that the total membership of the Committee does not exceed fourteen. The period of service on the Committee of a co-opted member shall end on or before the last day of the year in which that member was co-opted.

14. The Committee shall, from time to time, appoint an Ordinary member of the Society to be Editor of the Bulletin of the Society. The Editor of the Bulletin, whether or not a member of the Committee, shall be invited to attend all meetings of the Committee.

15. Any change to this Constitution by way of making a new Article or removing or amending any existing Article shall be valid if and only if the following procedure is followed:

(i) a proposal to change the Constitution is approved by at least seven members of the Committee at a meeting of the Committee;
(ii) the details of the said proposal are published in the Bulletin of the Society;
(iii) the said proposal is presented to a general meeting that takes place no sooner than one full calendar month after the publication of the said proposal in the Bulletin;
(iv) the said proposal is put to a vote and is approved by ten Ordinary members or at least two thirds of the Ordinary members present at the aforesaid meeting, whichever is the greater.

16. The Society shall make Rules for the regulation of the business of the Society. No Rule may conflict with any Article of this Constitution. A proposal to make, amend, or revoke a Rule or Rules may be made at any general meeting of the Society. Such a proposal shall be passed if and only if it is put to a vote and is approved by seven Ordinary members or at least two thirds of the Ordinary members present at that meeting, whichever is the greater.

17. Each motion submitted to a general meeting shall be passed if and only if it is approved by a simple majority of the members present provided that no Article of this Constitution stipulates otherwise.

RULES

Applying for membership

1. A person can make a proper application for election to Ordinary membership only by completing the appropriate application form (which can be obtained from any member of the Committee) and sending it to the Treasurer together with the subscription fee for one year.

2. An institution can make a proper application for election to Ordinary or Institutional membership only by completing the appropriate application form (which can be obtained from any member of the Committee) and sending it to the Treasurer together with the subscription fee for one year. The institution must name a person (or persons) who will act on its behalf in dealings with the Society.

3. The election of a person to Student membership may occur only at the last Committee meeting of a year. Any Student member elected shall be a member of the Society for the following year.

4. The Treasurer shall ensure that all proper applications for election to Ordinary or Institutional membership of the Society are decided upon by the Committee as laid down in the Constitution. The subscription fee paid by any candidate who is not elected to Ordinary or Institutional membership shall be returned to the candidate by the Treasurer.

5. The Treasurer shall have the discretion to add the name of any candidate for Ordinary or Institutional membership to the list of members as a provisional member until the Committee next meets.

Subscriptions

6. Every Ordinary and Institutional member shall pay during January in each year an annual subscription fee for that year. The current subscription fees are given below.

Institutional member	IR£50.00
Ordinary member	IR£10.00

7. The Committee shall have the discretion to end the membership of any Ordinary or Institutional member whose subscriptions are more than eighteen months in arrears.

Officers and Committee

8. The term of office of each Officer and the term of service of each other elected member of the Committee shall be two consecutive years starting on the first day of January that follows the meeting at which that Officer or member of the Committee was elected. The terms of office of the President and Vice-President shall start in an odd-numbered year; the terms of office of the Treasurer and Secretary shall start in an even-numbered year.

9. No person shall serve on the Committee for more than three terms consecutively. No President shall hold office for two consecutive terms.

10. The Committee shall have the discretion to appoint one of its members to take over the duties of any Officer of the Society who is unable to perform those duties; such an appointment shall end at or before the first general meeting that takes place after the appointment was made. If, at any general meeting, any Office should be vacant, then an election shall be held to fill that Office for the remainder of its current term.

11. The Committee shall form a sub-committee of at least four of its members, including two Officers, to manage the affairs of the Society between the times of Committee meetings.

12. The Committee shall appoint four Ordinary members of the Society to form an Editorial Board for the Bulletin of the Society under the chairmanship of The Editor of the Bulletin. This Board shall be responsible to the Committee for the publication of the said Bulletin.

13. The President shall be the chief executive officer of the Society. The normal duties of the President shall include chairing all meetings of the Society and of the Committee. The Vice-President shall act in lieu of the President as necessary.

14. The Secretary shall keep minutes of the meetings of the Society and of the Committee and shall issue notice of meetings to members resident in Ireland.

15. A financial statement for each year shall be written by the Treasurer holding office in that year, shall be duly audited by two persons appointed by the Committee, and shall be submitted to the first general meeting of the following year.

Minutes of the Meeting of the Irish Mathematical Society

Ordinary Meeting

8th April 1993

The Irish Mathematical Society held an ordinary meeting at 12.15 pm on Thursday 8th April 1993 at the DIAS, 10 Burlington Road. Eleven members were present. The president, B. Goldsmith, took the chair.

1. The **minutes** of the meeting of 22nd December 1992 were approved and signed.

2. There were no **matters arising**.

3. **Bulletin**
D. Tipple reported that the Bulletin is regularly delayed at the Eolas printers. The possibility of not using Eolas in future was discussed. This might mean cutting back on other expenditure. B. Goldsmith remarked that the Bulletin is seen by many as the main function of the Society. The Committee will look into the matter.

4. **IMS Constitution**
A new draft constitution (prepared by M. Ó Searcóid and D. Tipple) will be circulated to all members before the next ordinary meeting. A. G. O'Farrell suggested that the main changes to the constitution be explained to members.

5. **EMS**
B. Goldsmith read out his reply to a letter from the EMS asking for the IMS's views on the EMS. A. G. O'Farrell felt that the EMS should pressure the EC to stop favouring research with a "payoff" in preference to basic research. He also felt that Ireland should get as large a representation on the EMS as possible. The lack of funding of basic research by the Irish Government was noted.

S. Dineen felt that some prominent members of the EMS had a very narrow view of what should be funded. He also expressed concern at the EMS lending its name to certain conferences solely to help them secure funding. R. Timoney felt that the EMS meeting of fourteen people in Budapest was unnecessarily expensive. It was generally felt that the EMS is a very political organization, with an East v. West split, and also a split between those countries favouring individual membership and those favouring institutional membership.

It was agreed that A. G. O'Farrell need not represent the IMS at the forthcoming EMS meeting in London.

6. **Treasurer's business**
The Treasurer's report was circulated.
S. Dineen felt that the very low membership fee for overseas mathematicians loses the IMS respect.
The IMS pays an annual membership fee of £200 to the EMS. This is in addition to individuals' fees. The IMS may try to renegotiate this at some stage.

7. **September meeting**
A preliminary notice of this year's IMS meeting in Cork will be *e*-mailed shortly.

8. There was no **other business**

The meeting closed at 1.10 pm.

Graham Ellis
University College
Galway

CONFERENCES AT UNIVERSITY COLLEGE DUBLIN September 1994

7th Annual Meeting of the Irish Mathematical Society

5–6 September 1994

Speakers: J. M. Anderson (London), P. M. Gauthier (Montreal), B. Goldsmith (DIT), A. J. O'Farrell (Maynooth), J. V. Pulé (UCD), R. Ryan (UCG).

Requests for accommodation should be submitted by 1 July, 1994.
Conference dinner on Monday 5 September, 1994.
Further information: S. Dineen, S. Gardiner (addresses below).

Polynomials and Holomorphic Functions on Infinite Dimensional Spaces

7–9 September, 1994

Further information: S. Dineen, P. Mellon, C. Boyd.

Tel:	+353 1 706 8242
	+353 1 706 8265
Fax:	+353 1 706 1196
email:	sdineen@irlearn.bitnet
	gardiner@irlearn.bitnet

TRACE-ZERO MATRICES AND POLYNOMIAL COMMUTATORS

T. J. Laffey and T. T. West

Let $\mathbf{F}$ denote a field and $M_n(\mathbf{F})$ the algebra of $n \times n$ matrices over the field $\mathbf{F}$. If $X \in M_n(\mathbf{F})$, $\operatorname{tr}(X)$ will denote the trace of the matrix X. A well known result of Albert and Muckenhoupt [1] states that if $\operatorname{tr}(X) = 0$ then there exist matrices A, $B \in M_n(\mathbf{F})$ such that X is the commutator of A and B,

$$X = [A, B] = AB - BA.$$

Let p denote a polynomial in $\mathbf{F}[x]$ of degree greater than or equal to one. The *Polynomial Commutator* of A and B relative to p is defined to be

$$p[A, B] = p(AB) - p(BA).$$

It is easy to check, by examining the eigenvalues, that $\operatorname{tr}(p[A, B])$ is always zero. The Albert-Muckenhoupt result states that if $X \in M_n(\mathbf{F})$ with $\operatorname{tr}(X) = 0$ then, for $p(x) = x$,

$$X = p[A, B],$$

for some $A, B \in M_n(\mathbf{F})$. We show that, if the field $\mathbf{F}$ has characteristic zero the Albert-Muckenhoupt result may be extended to general polynomials of degree greater than, or equal to, one.

Theorem. *Let $\mathbf{F}$ be a field of characteristic zero and let $p \in \mathbf{F}[x]$ have degree greater than or equal to one. If $X \in M_n(\mathbf{F})$ is of trace zero then there exist matrices A, $B \in M_n(\mathbf{F})$ such that*

$$X = p[A, B].$$

First we prove the following elementary

Lemma. *If $\mathbb{F}$ is a field of characteristic zero and $X \in M_n(\mathbb{F})$ is of trace zero then we can choose a basis of $\mathbb{F}^n$ such that, relative to this basis, X has zeros on its main diagonal.*

Proof: Since $\text{tr}(X) = 0$ and $\mathbb{F}$ is of characteristic zero, X is not a scalar matrix. Thus there exists a vector $v \in \mathbb{F}^n$ such that v and Xv are linearly independent.

Set $v_1 = v$, $v_2 = Xv$ and extend to a basis $v_1, v_2, \ldots, v_n$ of $\mathbb{F}^n$. Relative to this basis

$$X = [x_{ij}]_{n\times n} \qquad \text{with } x_{11} = 0.$$

Further the matrix

$$Y = [x_{ij}]_{(n-1)\times(n-1)} \qquad (2 \leq i, j \leq n)$$

has trace zero and the proof may be completed by induction.

Proof of Theorem: Since $\text{tr}(X) = 0$ we may take

$$X = [x_{ij}]_{n\times n} \quad \text{with } x_{ii} = 0 \qquad (1 \leq i \leq n).$$

Now

$$X = L - U,$$

where L is a lower triangular matrix, U is an upper triangular matrix and both have zeros on the main diagonal.

Let D be the diagonal matrix

$$D = \text{diag}(d_1, \ldots, d_n),$$

then $p(D)$ is the diagonal matrix

$$p(D) = \text{diag}(p(d_1), \ldots, p(d_n)),$$

and since $\mathbb{F}$ is an infinite field and the degree of p is greater than, or equal to, one, we may choose the d_i so that the $p(d_i)$ are distinct $(1 \leq i \leq n)$.

Then

$$\begin{aligned} X &= (L + p(D)) - (U + p(D)) \\ &= L_1 - U_1 \end{aligned}$$

where $L_1 = L + p(D)$ is lower triangular and $U_1 = U + p(D)$ is upper triangular. The diagonal entries of L_1 and U_1 are $p(d_i)$, $(1 \leq i \leq n)$, and since these have been chosen distinct, the matrices L_1, U_1 and $p(D)$ are all similar. Thus there exist invertible S, $T \in M_n(\mathbb{F})$ so that

$$\begin{aligned} X &= S^{-1}p(D)S - T^{-1}p(D)T, \\ &= p(S^{-1}DS) - p(T^{-1}DT). \end{aligned}$$

Taking $A = S^{-1}T$ and $B = T^{-1}DS$ gives

$$X = p(AB) - p(BA) = p[A, B] \qquad (*)$$

which completes the proof.

Remarks

1. The result does not remain true if the restriction that $\mathbb{F}$ is of characteristic zero be dropped.
2. It would be interesting to investigate the latitude in equation $(*)$, for fixed X and p, in the possible choices of A and B.

Reference

[1] A. A. Albert and B. Muckenhoupt, *On matrices of trace zero*, Michigan J. Math. 4 (1957), 1–3.

T. J. Laffey,
University College,
Dublin.

T. T. West,
Trinity College,
Dublin.

A NEW GEOMETRIC INEQUALITY

Mícheál Ó Searcóid

Abstract: We prove the conjecture that a triangle whose three vertices lie in the three sides of a larger triangle must have perimeter at least as large as that of one of the other small triangles which are created by its inscription there; we also give proofs of some related results.

Introduction

On reading the abstract above, one might suspect that the title of this article ought to have been followed by a question mark. Tom Laffey, who first drew my attention to this conjecture, proved below as Theorem 3, pointed out that it had been listed as an unsolved problem by Kazarinoff [1, p78] and that, had it been proved in the meantime, it would most likely have appeared in the compendious work [2] — where it is not included.

It might be of interest to TeX enthusiasts if I add a little personal note here before embarking on the proof. Unlike most problems we encounter in modern mathematics, questions of this sort can be settled quickly and almost with certainty by using a computer. It is therefore worthwhile to try this avenue before expending time on possibly futile mathematical calculations. Believe it or not, my choice of language for testing the hypothesis was Knuth's character drawing programme METAFONT. It turns out that a METAFONT programme for this type of task is shorter and cleaner than one written in a standard all-purpose programming language, and that, provided one is careful to avoid arithmetic overflow, it is also accurate and quick. My short programme tested 800 inner triangles, chosen with partial randomness, in each of 1,000 outer triangles, also chosen with partial randomness. The programme took no more than a few minutes to write. It made the 800,000

tests and failed to find a counterexample to the conjecture. Given the nature of the conjecture, in particular the continuity involved in it, this made it at least as likely to be true as Fermat's Last Theorem.

Notation If A, B and C are points in a plane, we shall use $\sigma(ABC)$ to denote the sum of the three lengths $|BC|$, $|CA|$ and $|AB|$. If ABC is a triangle, then ΔABC will denote its area; otherwise ΔABC should be understood to be 0.

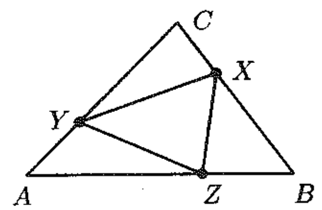
Perimeter theorems

It is not difficult to see that a proof of the truth of the conjecture will follow from a few technical manœuvres, and we shall demonstrate that such is the case, if we can establish first a related result, which we present now as Theorem 1.

Theorem 1. *Suppose ABC is a triangle and suppose X, Y and Z are points lying on the lines BC, CA and AB respectively. Then*

$$\max\{\sigma(AYZ), \sigma(XBZ), \sigma(XYC)\} \geq \frac{1}{2}\sigma(ABC)$$

with equality if and only if X, Y and Z are the mid-points of the line segments $[BC]$, $[CA]$ and $[AB]$ respectively.



Proof: We denote by a, b and c the lengths of the line segments $[BC]$, $[CA]$ and $[AB]$ respectively. When X, Y and Z are the mid-points described, the equality is well known. We assume, therefore, without loss of generality, that $|AZ| > \frac{1}{2}c$.

Now if $|CY| \leq \frac{1}{2}b$ and we denote the mid-points of $[AC]$ and $[AB]$ by M and N respectively, then we have

$$\sigma(AYZ) = |AM| + |MY| + |YZ| + |ZN| + |NA|$$
$$> |AM| + |MN| + |NA| = \frac{1}{2}\sigma(ABC)$$

and the required inequality follows. We may therefore assume that $|CY| > \frac{1}{2}b$. Similarly, we may assume that $|BX| > \frac{1}{2}a$. We now define r, s and t to be the strictly positive real numbers given by the equations

$$|BX| = \frac{1}{2}a + r, \qquad |CY| = \frac{1}{2}b + s, \qquad |AZ| = \frac{1}{2}c + t.$$

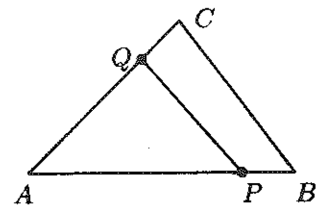
Now

$$\sigma(AYZ) \leq \frac{1}{2}\sigma(ABC)$$
$$\Rightarrow \frac{1}{2}c + t + \frac{1}{2}b - s + \sqrt{\left(\frac{1}{2}c+t\right)^2 + \left(\frac{1}{2}b - s\right)^2 - 2\left(\frac{1}{2}c+t\right)\left(\frac{1}{2}b-s\right)\cos A}$$
$$\leq \frac{1}{2}a + \frac{1}{2}b + \frac{1}{2}c$$
$$\Rightarrow \left(\frac{1}{2}c+t\right)^2 + \left(\frac{1}{2}b - s\right)^2 - 2\left(\frac{1}{2}c+t\right)\left(\frac{1}{2}b-s\right)\cos A$$
$$\leq \left(\frac{1}{2}a + s - t\right)^2$$
$$\Rightarrow \frac{1}{4}(b^2 + c^2 - 2bc\cos A) + ct - bs - (bt - cs - 2st)\cos A$$
$$\leq \frac{1}{4}a^2 + sa - ta - 2st$$
$$\Rightarrow (a+b+c)(t-s) \leq (bt - cs - 2st)(1 + \cos A)$$
$$\Rightarrow 2bc(t-s) \leq (bt - cs - 2st)(b + c - a)$$
$$\Rightarrow 0 \leq bt(b-c-a) + cs(b-c+a) - 2st(b+c-a)$$
$$\Rightarrow 0 \leq \frac{b}{s}(b-c-a) + \frac{c}{t}(b-c+a) - 2(b+c-a).$$

Similarly
$\sigma(XBZ) \leq \frac{1}{2}\sigma(ABC) \Rightarrow 0 \leq \frac{c}{t}(c-a-b) + \frac{a}{r}(c-a+b) - 2(c+a-b)$
and
$\sigma(XYC) \leq \frac{1}{2}\sigma(ABC) \Rightarrow 0 \leq \frac{a}{r}(a-b-c) + \frac{b}{s}(a-b+c) - 2(a+b-c)$.

That these three inequalities cannot be simultaneously satisfied is clear, because their addition would lead to the absurd $0 \leq -2(a+b+c)$. The theorem follows. □

Lemma 2. *Suppose ABC is a triangle and P and Q are points on the sides AB and AC respectively, with $|PB| = |QC| > 0$. Then $|PQ| < |BC|$.*



Proof: Set $|AB| = c$, $|CA| = b$, $|BC| = a$ and $|PQ| = d$.

Then

$$|BC|^2 - |PQ|^2$$
$$= (c^2 + b^2 - 2bc\cos A) -$$
$$((c-d)^2 + (b-d)^2 - 2(c-d)(b-d)\cos A)$$
$$= -2d^2 + 2cd + 2bd + 2(d^2 - bd - cd)\cos A$$
$$= 2d(b + c - d)(1 - \cos A) > 0,$$

and the lemma is proved. □

Theorem 3. *Suppose ABC is a triangle and X, Y and Z are points in the line segments $[BC]$, $[CA]$ and $[AB]$ respectively. Then*

$$\sigma(XYZ) \geq \min\{\sigma(AYZ), \sigma(XBZ), \sigma(XYC)\}$$

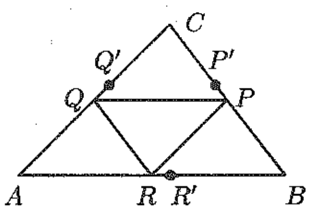
with equality if and only if X, Y and Z are the mid-points of the line segments $[BC]$, $[CA]$ and $[AB]$ respectively.

Proof: The function f defined on the compact set $[BC]\times[CA]\times[AB]$ by

$$f(X,Y,Z)=\sigma(XYZ)-\min\{\sigma(AYZ),\sigma(XBZ),\sigma(XYC)\}$$

is continuous and therefore attains its minimum value. Suppose this minimum value is attained at (P,Q,R). Since f has the value 0 when X, Y and Z are the mid-points of the line segments $[BC]$, $[CA]$ and $[AB]$ respectively, we have $f(P,Q,R)\leq 0$. It is easily verified that if any of X, Y or Z coincides with any of A, B or C then $f(X,Y,Z)>0$. It follows that P, Q and R are internal points of their respective line segments.

We want to establish firstly that the three quantities $\sigma(AQR)$, $\sigma(PBR)$ and $\sigma(PQC)$ are equal. To this end, we accept, without loss of generality, that $\sigma(AQR)\leq\sigma(PBR)\leq\sigma(PQC)$.



Suppose now that $\sigma(AQR)<\sigma(PBR)$ and consider an internal point R' of the line segment $[BR]$ which is close enough to R to ensure that $\sigma(AQR')<\sigma(PBR')$. Note that it is true in any case that $\sigma(PBR')<\sigma(PBR)\leq\sigma(PQC)$.

So we have

$$\begin{aligned}&f(P,Q,R')\\&\quad=\sigma(PQR')-\sigma(AQR')\\&\quad=\sigma(PQR)-\sigma(AQR)-(|PR|+|RR'|-|PR'|)\\&\quad<\sigma(PQR)-\sigma(AQR)\\&\quad=f(P,Q,R)\end{aligned}$$

contradicting the minimality of f at (P,Q,R). We must therefore infer that $\sigma(AQR)=\sigma(PBR)$.

Suppose now that $\sigma(AQR)<\sigma(PQC)$ and consider internal points P' and Q' of the line segments $[PC]$ and $[QC]$ respectively, with $|PP'|=|QQ'|$ and this quantity being small enough to ensure that the inequalities $\sigma(AQ'R)<\sigma(P'Q'C)$ and $\sigma(P'BR)<\sigma(P'Q'C)$ hold. We note that Lemma 2 implies that $|PQ|>|P'Q'|$.

Then

$$\begin{aligned}\sigma(P'Q'R)-\sigma(P'BR)&=\sigma(PQR)-\sigma(PBR)+|Q'R|+|P'Q'|\\&\quad-|QR|-|PQ|-|PP'|\\&=\sigma(PQR)-\sigma(PBR)-(|PQ|-|P'Q'|)\\&\quad-(|Q'Q|+|QR|-|Q'R|)\\&<\sigma(PQR)-\sigma(PBR)\\&=f(P,Q,R).\end{aligned}$$

Similarly

$$\begin{aligned}\sigma(P'Q'R)-\sigma(AQ'R)&<\sigma(PQR)-\sigma(AQR)\\&=f(P,Q,R).\end{aligned}$$

It follows that $f(P',Q',R)<f(P,Q,R)$, so that minimality of f at (P,Q,R) is once again contradicted. We must therefore have $\sigma(PBR)=\sigma(AQR)=\sigma(PQC)=q$, say.

Now

$$\begin{aligned}q\geq\sigma(PQR)&=\sigma(AQR)+\sigma(PBR)+\sigma(PQC)-\sigma(ABC)\\&=3q-\sigma(ABC),\end{aligned}$$

so that $q\leq\frac{1}{2}\sigma(ABC)$. It follows from Theorem 1 that equality holds and that P, Q and R are the mid-points of the line segments $[BC]$, $[CA]$ and $[AB]$ respectively, and the theorem is proven. □

Area theorems

One might expect area analogues to the perimeter theorems of the last section, and one should be right to do so. Indeed, the analogue of Theorem 3 was known to Kazarinoff [1], though he simply states that it is true without giving a reference to a proof.

Theorem 4. *Suppose ABC is a triangle and X, Y and Z are interior points of the line segments [BC], [CA] and [AB] respectively. Then*

$$\Delta XYZ \geq \min\{\Delta AYZ, \Delta XBZ, \Delta XYC\}$$

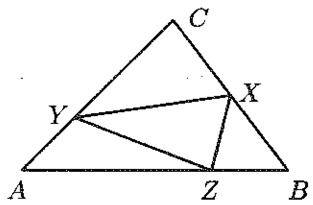
with equality if and only if X, Y and Z are the mid-points of the line segments [BC], [CA] and [AB] respectively.

A proof of Theorem 4 can be effected rather easily by setting up a function which attains its bound and manipulating perturbations, only provided the analogue of Theorem 1 has first been established. That analogue, given below as Theorem 5, is much easier to demonstrate than Theorem 1. It is left to the reader to supply a proof of Theorem 4 using Theorem 5.

Theorem 5. *Suppose ABC is a triangle and suppose X, Y and Z are points lying on the lines BC, CA and AB respectively. Then*

$$\min\{\Delta AYZ, \Delta XBZ, \Delta XYC\} \leq \frac{1}{4}\Delta ABC$$

with equality if and only if X, Y and Z are the mid-points of the line segments [BC], [CA] and [AB] respectively.



Proof: Evidently

$$\Delta AYZ \geq \frac{1}{4}\Delta ABC \Rightarrow 4|AY||AZ| \geq bc$$

$$\Delta XBZ \geq \frac{1}{4}\Delta ABC \Rightarrow 4|BZ||BX| \geq ca$$

and

$$\Delta XYC \geq \frac{1}{4}\Delta ABC \Rightarrow 4|CX||CY| \geq ab,$$

where a, b and c denote the lengths of the line segments $[BC]$, $[CA]$ and $[AB]$ respectively.

Multiplying the three inequalities at the right, we get

$$4|AZ||ZB| \times 4|AY||YC| \times 4|CX||XB| \geq a^2b^2c^2.$$

Since $|AZ| + |ZB| = c$, $|AY| + |YC| = b$ and $|CX| + |XB| = a$, it follows that the three inequalities at the left can be simultaneously satisfied only if X, Y and Z are the mid-points of their respective sides. □

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Mícheál Ó Searcóid,
Roinn na Matamaitice,
Coláiste na hOllscoile,
Baile Átha Cliath.

WHAT'S THE PROBABILITY OF GENERATING A CYCLIC SUBGROUP?

D. M. Patrick, G. J. Sherman, C. A. Sugar and E. K. Wepsic

1. Introduction

A recent coffee-room conversation at Rose-Hulman about an old number theory gem — two randomly chosen integers are relatively prime with probability $6/\pi^2$ — led to the following exchange.

A group theorist: "You know, $6/\pi^2$ sounds a lot like the 5/8 bound for commutativity to me."

$\mathrm{Pr_2Comm}(G) = \dfrac{|\{(x,y) \in G^2 \mid xy = yx\}|}{|G|^2}$ is either one or at most 5/8 for finite groups [2].

A topologist: "Oh no! What are you going to do, turn that one into a group theory problem too?"

Here's a try: If x and y are group elements instead of integers, then "are relatively prime" should mean there does not exist an element g in the group such that g "divides" both x and y. Unfortunately (at least for this interpretation) the equations $gz = x$ and $gz = y$ each have solutions for any g in the group. Another try — 6 and 9 are not relatively prime because they generate a proper (cyclic, of course) subgroup of the integers, — suggests the title of this paper.

More formally, let G be a finite group and set

$$\mathrm{Pr_2Cyc}(G) = \frac{|\{(x,y) \in G^2 \mid \langle x,y\rangle \text{ is cyclic}\}|}{|G|^2}.$$

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If $\langle x,y\rangle$ is cyclic, we say the ordered pair (x,y) is cyclic. The purpose of this note is to show that if G is not cyclic, then $\mathrm{Pr_2Cyc}(G) \le 5/8$. A generalization to cyclic n-tuples — an n-tuple $(x_1, x_2, \ldots, x_n)$ for which $\langle x_1, x_2, \ldots, x_n\rangle$ is cyclic — is also established.

It is well known that the 5/8 bound for commutativity can be replaced by $(p^2+p-1)/p^3$ where p is the smallest prime dividing the order of G. Rewriting our results in terms of p is left as an exercise for the reader.

2. Cyclic Ordered Pairs

Theorem 1. $\mathrm{Pr_2Cyc}(G) = 1$ *if G is cyclic; in every other case,* $\mathrm{Pr_2Cyc}(G) \le 5/8$.

Our proof is woven from the 5/8 bound for commutativity and a sequence of lemmas. Suppose firstly that G is non-abelian. Then, since two elements that generate a cyclic subgroup must commute,

$$\mathrm{Pr_2Cyc}(G) \le \mathrm{Pr_2Comm}(G) \le 5/8$$

by the result of [2]. And fortunately,

Lemma 1. $\mathrm{Pr_2Cyc}(G) = 1$ *if, and only if, G is cyclic.*

Proof: If G is cyclic, certainly each subgroup of G is cyclic; i.e., $\mathrm{Pr_2Cyc}(G) = 1$. On the other hand, if $\mathrm{Pr_2Cyc}(G) = 1$, then G is certainly abelian and $\mathrm{Pr_2Cyc}(S_p) = 1$ for each p-Sylow subgroup of G. This means each p-Sylow subgroup is cyclic and, therefore, that G is cyclic.

Now we may restrict our attention to non-cyclic abelian groups.

Lemma 2. *If $G \cong H \oplus K$, then*

$$\mathrm{Pr_2Cyc}(G) \le \mathrm{Pr_2Cyc}(H) \cdot \mathrm{Pr_2Cyc}(K);$$

i.e., $\mathrm{Pr_2Cyc}(G)$ is submultiplicative.

Proof: If the pair $((x_1,y_1),(x_2,y_2))$ is cyclic, then there exists (x,y) in $H \oplus K$ and there exist non-negative integers s_1 and s_2 such that $(x,y)^{s_i} = (x_i,y_i)$; i.e., both (x_1,x_2) and (y_1,y_2) are cyclic.

In view of Lemma 2,

$$\mathrm{Pr_2Cyc}(G) \leq \prod_{p||G|} \mathrm{Pr_2Cyc}(S_p).$$

where p denotes a prime and S_p is the p-Sylow subgroup of G. Thus, if $\mathrm{Pr_2Cyc}(S_p) \leq 5/8$ for at least one p, the theorem follows.

Since G is non-cyclic, there exists at least one prime, say q, for which S_q is non-cyclic. This means that S_q is of the form $\mathbf{Z}_{q^k} \oplus \mathbf{Z}_{q^m} \oplus A$ with $1 \leq k \leq m$.

Lemma 3. $\mathrm{Pr_2Cyc}(\mathbf{Z}_{q^k} \oplus \mathbf{Z}_{q^k}) \leq 5/8$.

Proof: Let $((a,b),(c,d))$ be a cyclic ordered pair in $\mathbf{Z}_{q^k} \oplus \mathbf{Z}_{q^k}$. We proceed by cases.

Case: The order of (a,b) is q^k. Since q^k is the maximum order of an element of $\mathbf{Z}_{q^k} \oplus \mathbf{Z}_{q^k}$, it follows that $(c,d) \in \langle(a,b)\rangle$. Thus there are $q^{2k} - q^{2k-2}$ choices for (a,b) and q^k choices for (c,d); i.e., there are $(q^{2k} - q^{2k-2})q^k$ choices for $((a,b),(c,d))$.

Case: The order of (a,b) is less than q^k. The number of choices for (a,b) is q^{2k-2} and the number of choices for (c,d) is certainly bounded above by q^{2k}; i.e., there are at most $q^{2k-2} \cdot q^{2k}$ choices for $((a,b),(c,d))$.

Therefore,

$$\begin{aligned}\mathrm{Pr_2Cyc}(\mathbf{Z}_{q^k} \oplus \mathbf{Z}_{q^k}) &\leq \frac{(q^{2k} - q^{2k-2})q^k + q^{2k-2} \cdot q^{2k}}{q^{4k}} \\ &= \frac{1}{q^2} + \left(1 - \frac{1}{q^2}\right)\left(\frac{1}{q^k}\right) \\ &\leq \frac{1}{4} + \frac{3}{4} \cdot \frac{1}{2} \\ &= \frac{5}{8}.\end{aligned}$$

It is easy to check that $\mathrm{Pr_2Cyc}(\mathbf{Z}_2 \oplus \mathbf{Z}_2) = 5/8$.

Lemma 4. $\mathrm{Pr_2Cyc}(\mathbf{Z}_{q^k} \oplus \mathbf{Z}_{q^m}) \leq 5/8$ *implies that* $\mathrm{Pr_2Cyc}(\mathbf{Z}_{q^k} \oplus \mathbf{Z}_{q^{m+1}}) \leq 5/8$.

Proof: Let $((a,b),(c,d))$ be a cyclic ordered pair in $\mathbf{Z}_{q^k} \oplus \mathbf{Z}_{q^{m+1}}$. Again, we proceed by cases.

Case: $|(a,b)|, |(c,d)| \leq m$. Let's collect all such cyclic ordered pairs in a set, say $C = \{((a,b),(c,d)) \mid |(a,b)|, |(c,d)| \leq m\}$, and collect the components of elements of C in a set, say $H = \bigcup_C \{(a,b),(c,d)\}$. Now denoting the projection of H onto $\mathbf{Z}_{q^{m+1}}$ by B, we have $H \subseteq \langle A \rangle \oplus \langle B \rangle$. Thus $\langle B \rangle$ is a cyclic subgroup of $\mathbf{Z}_{q^{m+1}}$ containing no elements of order q^{m+1}. Therefore $\langle B \rangle \cong \mathbf{Z}_{q^j}$, where $j \leq m$, which implies that $\langle A \rangle \oplus \langle B \rangle$ is isomorphic to a subgroup of $\mathbf{Z}_{q^n} \oplus \mathbf{Z}_{q^m}$. Our inductive hypothesis yields $|C| \leq (5/8)q^{2(k+m)}$.

Case: $|(a,b)| = q^{m+1}$. There are $q^{k+m+1} - q^{k+m}$ choices for (a,b) and q^{m+1} choices for (c,d) since $(c,d) \in \langle(a,b)\rangle$. Therefore, there are $q^{k+2m+2} - q^{k+2m+1}$ choices for $((a,b),(c,d))$.

Case: $|(a,b)| \leq q^m$ and $|(c,d)| = q^{m+1}$. There are $q^{k+m+1} - q^{k+m}$ choices for (c,d) and q^m choices for (a,b) since $(a,b) \in \langle(c,d)\rangle$ and has order less than q^{m+1}. Therefore, there are $q^{k+2m+1} - q^{k+2m}$ choices for $((a,b),(c,d))$.

Now we have

$$\begin{aligned}\mathrm{Pr_2Cyc}(\mathbf{Z}_{q^k} \oplus \mathbf{Z}_{q^{m+1}}) &\leq \frac{(5/8)q^{2(k+m)} + q^{k+2m+2} - q^{k+2m}}{q^{2(k+m+1)}} \\ &= \frac{5}{8} \cdot \frac{1}{q^2} + \frac{1}{q^k} - \frac{1}{q^{k+2}} \\ &\leq \frac{5}{8} \cdot \frac{1}{2^2} + \frac{1}{2^k} - \frac{1}{2^{k+2}} \\ &\leq \frac{5}{8} \cdot \frac{1}{2^2} + \frac{1}{2} - \frac{1}{2^3} \\ &= \frac{17}{32}.\end{aligned}$$

Therefore $\mathrm{Pr_2Cyc}(S_q) \leq 5/8$ and the proof of the theorem is complete.

3. Cyclic n-tuples

Does the theorem generalize to cyclic n-tuples? It hinges on the availability of a 5/8-like bound for $\mathrm{Pr}_n\mathrm{Comm}(G)$, the proportion of mutually commutative n-tuples ($x_i x_j = x_j x_i$ for all i and j) in G. Erdös and Strauss [1] established a lower bound for

$\Pr_n\text{Comm}(G)$ but the following upper bound does not appear to be well-known. Indeed, to the best of our knowledge, this upper bound has appeared only in an unpublished note (Testing Laws in Groups) of John D. Dixon's which circulated in the late 1970's. We include it here with our elementary proof.

Lemma 5. *If G is nonabelian, then*

$$\Pr_n\text{Comm}(G) \le \frac{3}{2^n} - \frac{1}{2^{2n-1}}.$$

Proof: Given the 5/8 bound for $n = 2$, one observes that either the first component of a mutually commutative n-tuple is in the center, $Z = Z(G)$, of G (with probability $|Z|/|G|$) or it isn't (with probability $1 - |Z|/|G|$). Thus,

$$\begin{aligned}
\Pr_n\text{Comm}(G) &\le \frac{|Z|}{|G|}\cdot\left(\frac{3}{2^{n-1}} - \frac{1}{2^{2n-3}}\right) + \left(1 - \frac{|Z|}{|G|}\right)\cdot\frac{1}{2^{n-1}} \\
&= \frac{|Z|}{|G|}\cdot\left(\frac{3}{2^{n-1}} - \frac{1}{2^{2n-3}} - \frac{1}{2^{n-1}}\right) + \frac{1}{2^{n-1}} \\
&\le \frac{1}{4}\left(\frac{3}{2^{n-1}} - \frac{1}{2^{2n-3}} - \frac{1}{2^{n-1}}\right) + \frac{1}{2^{n-1}} \\
&= \frac{3}{2^n} - \frac{1}{2^{2n-1}},
\end{aligned}$$

because $|Z|/|G| \le 1/4$ and each component must be in the centralizer of the first component. We remark that $\Pr_n\text{Comm}(G)$ assumes the bound if, and only if, $G/Z \cong \mathbf{Z}_2 \oplus \mathbf{Z}_2$.

With Lemma 5 in hand one can generalize the proof of the Theorem 1 to cyclic n-tuples by replacing each occurrence of 5/8 with $\frac{3}{2^n} - \frac{1}{2^{2n-1}}$:

Theorem 2. $\Pr_n\text{Cyc}(G)$ *is either one or it is at most* $\frac{3}{2^n} - \frac{1}{2^{2n-1}}$.

Acknowledgement.

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D. M. Patrick,
Department of Mathematics,
Massachusetts Institute of Technology,
Cambridge, MA 02139,
USA.

G. J. Sherman,
Department of Mathematics,
Rose-Hulman Institute of Technology,
Terre Haute, IN 47803,
USA.

C. A. Sugar,
Department of Mathematics,
Stanford University,
Stanford, CA 94305,
USA.

E. K. Wepsic,
Department of Mathematics,
Harvard University,
Cambridge, MA 02138,
USA.

FUNCTION ALGEBRAS

Gerard J. Murphy

1. Introduction

The theory of function algebras forms an important branch of functional analysis. Its central problem is to determine whether a given complex-valued function can be uniformly approximated by the elements of a prescribed algebra of functions. One of the attractive features of the subject is that it involves a beautiful interplay of ideas and methods from a variety of sources, such as topology and algebra, and especially from functional analysis and the theory of analytic functions. Moreover, it has important applications, for instance, to classical analysis and to operator theory. Indeed, the concepts and techniques of function algebra theory often produce new insights into the classical theory of approximation by analytic functions, and raise new questions, which serve to enliven and reinvigorate that subject.

The theory of function algebras is so extensive that any short account must be selective, and this is the case for the present exposition. The intention here is to explain some of the core ideas and problems, and to give an account of an aspect of the theory of particular interest to the author. This aspect is the theory of generalized Hardy H^p spaces, and it is of interest in operator theory because of its applications to the theory of Toeplitz operators. Part of its importance in the theory of function algebras relates to one of the major problems of the subject, namely, the determination of conditions under which it is possible to embed analytic structure into the spectrum of a function algebra.

The paper is organized as follows: In Section 2 we discuss the basic concepts and give an illustration of one of these concepts,

that of a representing measure, to prove a maximality theorem of Wermer. In Section 3, we discuss some important classes of function algebras. In Section 4 we give a brief account of generalized H^p space theory and indicate how this theory is applied to the problem of embedding analytic structure. Finally, in Section 5, we discuss some connections with operator theory.

2. Representing measures

A *function algebra* on a compact Hausdorff space Ω is a closed subalgebra A of the algebra $C(\Omega)$ of all continuous functions on Ω, that contains the constants and separates the points of Ω. (The operations on $C(\Omega)$ are the usual pointwise-defined ones and the norm is the supremum norm, given by $\|\varphi\| = \sup_{s\in\Omega} |\varphi(s)|$.)

Of course, $C(\Omega)$ is a function algebra, but it is not typical of the algebras that are of primary interest to function algebraists. Rather, the prototypical function algebra is the disc algebra. This is the set $\mathbf{A}$ of all continuous functions on the closed unit disc $\mathbf{D}$ that are analytic in the interior, and it is easily seen that $\mathbf{A}$ is indeed a function algebra. This algebra can also be realized as a function algebra on the unit circle $\mathbf{T}$, because the homomorphism from $C(\mathbf{D})$ to $C(\mathbf{T})$ got by sending a function on $\mathbf{D}$ to its restriction to $\mathbf{T}$ induces an isometric algebra isomorphism of $\mathbf{A}$ onto the closed subalgebra of $C(\mathbf{T})$ generated by the inclusion function $z\colon \mathbf{T} \to \mathbf{C}$ (always, z will denote this function). It is usual to refer to $\mathbf{A}$ as the disc algebra on the disc and to its image on $\mathbf{T}$ as the disc algebra on the circle. It follows easily from Fejér's theorem that the disc algebra on $\mathbf{T}$ is the set of all elements of $C(\mathbf{T})$ whose Fourier transform is supported in the set $\mathbf{Z}^+$ of all non-negative integers.

We defer giving more examples of function algebras to the next section. Instead, we introduce the concept of a representing measure, one of the most important ideas of the theory.

Let A be a function algebra on a compact Hausdorff space Ω. If τ is a bounded linear functional on A, then, by the Hahn–Banach theorem and the Riesz–Kakutani representation theorem, there is a regular Borel complex measure m on Ω, called a *representing measure* for τ, such that

$$\tau(\varphi) = \int \varphi\, dm \qquad (\varphi \in A)$$

and $\|\tau\| = \|m\|$, where $\|m\|$ is the total variation norm, $\|m\| = |m|(\Omega)$. (In general, the representing measure m is not unique.) The case of most interest for the theory of function algebras is that in which τ is a *character* of A, that is, a non-zero multiplicative linear functional on A. Characters are automatically of norm one and map the unit to 1, from which it follows that any representing measure of a character is positive and of total mass one, that is, a probability measure.

A point concerning notation is needed before we proceed further: If $g \in C(\Omega)$ and μ is a regular Borel complex measure on Ω, we denote by $g\mu$ the regular Borel complex measure on Ω corresponding to the bounded linear functional on $C(\Omega)$ given by $f \mapsto \int fg\, d\mu$.

To illustrate the idea of a representing measure, let us consider again the disc algebra $\mathbf{A}$ on the circle. Let m be normalized Lebesgue measure on $\mathbf{T}$, so

$$\int \varphi\, dm = \frac{1}{2\pi}\int_0^{2\pi} \varphi(e^{it})\, dt \qquad (\varphi \in C(\mathbf{T})).$$

Since $\int z^n\, dm = 0$ for all $n > 0$, it follows that $\int \varphi z\, dm = 0$ for all $\varphi \in A$. Now let s be a point in the unit disc $\mathbf{D}$. Then s defines a character τ_s on A by setting $\tau_s(\varphi) = \varphi(s)$. (Recall that $\mathbf{A}$ is the isomorphic image of the disc algebra on $\mathbf{D}$. We are using this to identify elements of $\mathbf{A}$ with their corresponding extensions to $\mathbf{D}$.) Suppose now that $|s| < 1$. If $\varphi \in \mathbf{A}$, then $\varphi = \varphi(s) + (z - s)\psi$, for some $\psi \in \mathbf{A}$. Setting $\mu = (1 - \bar{z}s)^{-1}m$, we have $\int 1\, d\mu = \int \sum_{n=0}^{\infty}(s\bar{z})^n\, dm = \sum_{n=0}^{\infty} s^n \int \bar{z}^n\, dm = 1$, so $\int \varphi\, d\mu = \varphi(s) + \int \psi z\, dm = \varphi(s)$. This is, of course, the Cauchy integral formula. The norm of the measure μ is not equal to 1 if s is non-zero, so μ is not a representing measure for τ_s, but with a little further manipulation we get a representing measure: Since $(1 - \bar{s}z)^{-1}$ belongs to $\mathbf{A}$, therefore

$$\frac{\varphi(s)}{1 - |s|^2} = \int \frac{\varphi}{1 - \bar{s}z}\, d\mu = \int \frac{\varphi}{|1 - s\bar{z}|^2}\, dm,$$

so $\varphi(s) = \int \varphi\, dm_s$, where m_s is the probability measure

$$m_s = \frac{1 - |s|^2}{|1 - s\bar{z}|^2} m.$$

Thus, m_s is a representing measure for τ_s. The equation $\varphi(s) = \int \varphi\, dm_s$ is the Poisson integral formula.

It is easily seen that for any character τ on $\mathbf{A}$, there is only one representing measure for τ (see Section 3). We shall now give an application of this to prove the following theorem due to Wermer (the proof is due to Hoffman and Singer). First, recall an elementary fact from Banach algebra theory: If B is a unital abelian Banach algebra, then an element $b \in B$ is invertible if and only if $\tau(b) \neq 0$ for all characters τ of B.

2.1. Theorem. *If B is a function algebra on the circle $\mathbf{T}$ containing the disc algebra $\mathbf{A}$, then either $B = \mathbf{A}$ or $B = C(\mathbf{T})$.*

Proof. First suppose that $\tau(z) \neq 0$ for all characters τ of B. Then z is invertible in B, and therefore $\bar{z} = 1/z \in B$, so by Fejér's theorem, $B = C(\mathbf{T})$. Suppose on the other hand that for some character τ of B, we have $\tau(z) = 0$. Then $\tau(\varphi) = \varphi(0)$ for all $\varphi \in \mathbf{A}$, so if μ is a representing measure for τ (as a character of B), then for all $\varphi \in \mathbf{A}$ we have $\int \varphi\, dm = \varphi(0) = \int \varphi\, d\mu$, and therefore $\mu = m$, by uniqueness of representing measures for $\mathbf{A}$. Hence, for all $\varphi \in B$ and for all $n > 0$, we have $\int \varphi z^n\, dm = \tau(\varphi z^n) = \tau(\varphi)\tau(z)^n = 0$. Thus, the Fourier transform of φ is supported in $\mathbf{Z}^+$, and therefore $\varphi \in \mathbf{A}$. Hence, $B = \mathbf{A}$. □

This maximality result for $\mathbf{A}$ has some interesting consequences. One of them is that there are many continuous functions that are boundary values of analytic functions. Specifically, let B be the set of all $\varphi \in C(\mathbf{T})$ for which there exists an analytic function ψ on the open unit disc such that for almost all t in $[0, 2\pi)$, $\psi(\lambda)$ converges to $\varphi(e^{it})$ as λ converges to e^{it} non-tangentially. Then B is a subalgebra of $C(\mathbf{T})$ containing $\mathbf{A}$. It can be shown that the containment is proper, so by Wermer's theorem, B is dense in $C(\mathbf{T})$. For details see [11].

Another application of the maximality theorem asserts that if F is a proper closed subset of $\mathbf{T}$, then every continuous function on F can be uniformly approximated by polynomials (*loc. cit.*).

The concept of a representing measure may perhaps seem, at first sight, not to be a very significant idea. The preceding examples and applications help to give some inkling of its power, more particularly in the case of a unique representing measure for a character. In fact, in the latter setting one can generalize a large amount of the classical theory of H^p spaces. Moreover, this generalization has useful applications, one of which we shall see in Section 4.

3. Dirichlet and logmodular algebras

Let K be a non-empty compact subset of $\mathbf{C}^n$. There are a number of important function algebras associated with K: The algebra $A(K)$ is defined to be the set of all continuous functions on K that are holomorphic on the interior of K, and the algebra $R(K)$ is defined to be the set of all continuous functions on K that are uniformly approximable by rational functions with no poles on K. By $P(K)$ we denote the algebra of all continuous functions on K which are uniformly approximable by polynomials. Clearly, $P(K) \subseteq R(K) \subseteq A(K) \subseteq C(K)$, and a part of the theory of function algebras is given over to the problem of determining when one has equality at some point in this chain of inclusions.

Another significant class of function algebras is formed by the Arens–Singer algebras, which are generalizations of the disc algebra on the circle. Let G be a non-trivial subgroup of the additive group $\mathbf{R}$, endowed with the discrete topology, so that the Pontryagin dual group $\hat{G}$ is therefore compact. Each element $x \in G$ defines a continuous character ε_x on $\hat{G}$ by evaluation. Denote by $\mathbf{A(G)}$ the closed linear span in $C(\hat{G})$ of the characters ε_x ($x \in G^+$), where $G^+ = \mathbf{R}^+ \cap G$. Then $\mathbf{A(G)}$ is a function algebra on $\hat{G}$. It can be shown to be isomorphic to an algebra of analytic almost-periodic functions in the upper half-plane, see [10], for example.

As the theory of function algebras developed, analyticity, at least in residual form, seemed to pervade the subject, and a natural question presented itself, namely, if A is a function algebra on Ω, and $A \neq C(\Omega)$, to what extent do the functions in A behave like analytic functions? One observes that, for example, there is a shortage of real-valued functions among analytic functions, and a shortage of real-valued functions persists whenever $A \neq C(\Omega)$, as a consequence of the Stone–Weierstrass theorem ($A = C(\Omega)$ if and only if the real functions in A separate the points of Ω). Moreover, many analytic-type phenomena, such as Jensen's inequality and the maximum modulus principle, were observed to appear in great generality. Indeed, it was discovered that genuine analyticity existed in situations of a very general character.

We shall make this a little more precise: If A is a function algebra on Ω, then one can embed Ω homeomorphically into the spectrum X of A. (As a set, X consists of the characters of A and it is made into a compact Hausdorff space by endowing it with the relative weak* topology induced from the dual space A^*.) It is easily seen that if τ_ω is defined by $\tau_\omega(\varphi) = \varphi(\omega)$, then the map

$$\Omega \to X, \quad \omega \mapsto \tau_\omega,$$

is a homeomorphism of Ω onto a subspace of X. At one time it was conjectured that whenever X is larger than Ω, there had to be some analytic structure in X, in the sense that there should be an embedding θ of a disc into X such that for all $\varphi \in A$, the functions $\hat{\varphi} \circ \theta$ are analytic. (By $\hat{\varphi}$ we denote the Gelfand transform of φ, that is, the continuous function on Ω defined by $\hat{\varphi}(\tau) = \tau(\varphi)$.) Support for this conjecture came from a remarkable theorem of Wermer, concerning embedding analytic structure in a certain very large class of function algebras. We shall discuss this result in more detail presently. However, in 1963, G. Stolzenberg gave an example which falsified the conjecture and Garnett later exhibited examples that showed that the spectrum could, in a sense, be quite arbitrary. We shall shortly make this more precise. First, we discuss some positive results.

Again suppose that A is a function algebra on Ω. Define the relation $\sim$ on the spectrum X of A by $\tau \sim \sigma$ if $\|\tau - \sigma\| < 2$. This relation is an equivalence relation on X. The search for analytic structure in the spectrum has been conducted in terms of the corresponding equivalence classes, called *Gleason parts*.

We now consider a large class of function algebras, first introduced by A. M. Gleason, called *Dirichlet* algebras. A function algebra A on Ω is such an algebra if every real-valued continuous function on Ω can be uniformly approximated by real parts of functions in A.

Of course, $C(\Omega)$ is trivially a Dirichlet algebra, but these are not the interesting examples. The prototype is, as usual in this subject, the disc algebra $\mathbf{A}$. It follows easily from the density of the set of trigonometrical polynomials in $C(\mathbf{T})$ that $\mathbf{A} + \bar{\mathbf{A}}$ is dense in $C(\mathbf{T})$, and hence that $\mathbf{A}$ is a Dirichlet algebra ($\bar{\mathbf{A}}$ denotes the set of complex conjugates of elements of $\mathbf{A}$). It is an immediate consequence of the definition that a representing measure for a character on a Dirichlet algebra is unique, so in particular, this applies to $\mathbf{A}$, and proves the uniqueness claim we made in Section 2.

The Arens–Singer algebras $\mathbf{A}(\mathbf{G})$ are Dirichlet algebras, for the same reason as in the case of the disc algebra, namely density of the (generalized) trigonometric polynomials.

If K is a compact subset of the plane whose complement is connected, and if ∂K is the boundary of K, then $P(\partial K)$ is a Dirichlet algebra on ∂K [3].

If A denotes the closure in $C(\mathbf{T}^2)$ of the trigonometric polynomials $\varphi = \sum_{n=0}^{N} \sum_{m=0}^{N} \lambda_{nm} z_1^n z_2^m$, then A is a function algebra on $\mathbf{T}^2$ that is not a Dirichlet algebra.

Now suppose that S is a subset of the spectrum X of a function algebra A. If S can be given the structure of a connected complex manifold in such a way that the functions in A, or rather their Gelfand transforms, are analytic on S, then it can be shown that S must lie entirely in a single Gleason part. The embedding theorem of Wermer referred to earlier in this section asserts that if A is a Dirichlet algebra, then for each Gleason part of X that is not a singleton, there a homeomorphism θ of the open unit disc onto the part, where the latter is endowed with the metric (norm) topology, such that $\hat{\varphi} \circ \theta$ is analytic for all $\varphi \in A$. The proof of this theorem involves generalized Hardy space theory and a sketch of the method of proof is given in Section 4.

Wermer's result was extended by K. Hoffman to a more general class of function algebras, namely to logmodular algebras. A function algebra A on Ω is *logmodular* if every real-valued continuous function φ on Ω can be uniformly approximated by elements of the form $\log|\psi|$, where ψ is an invertible element of A. The equation $\mathrm{Re}(\varphi) = \log|e^{\varphi}|$ shows that Dirichlet algebras are logmodular, but the converse is false, as we shall see in Section 4. Logmodular algebras share an important property with Dirichlet algebras, namely, uniqueness of representing measures for characters.

Despite these positive results, Stolzenberg's example shows that unless restrictive hypotheses are imposed, analytic structure may not be present in the Gleason parts of a function algebra. Indeed, not much can be said about the structure of Gleason parts in general, as can be seen from the following result of Garnett [8]: Given any completely regular σ-compact space Y there exists a function algebra in whose spectrum Y can be embedded as a single Gleason part.

4. Generalized $\mathbf{H}^{\mathbf{p}}$ space theory

We motivate our considerations in this section by briefly considering the classical Hardy space theory. If p is a real number not less than 1, the Hardy space H^p is defined to be the set of all analytic functions f on the open unit disc for which $\sup_{0\leq r<1}\int_0^{2\pi}|f(re^{it})|^p\,dt$ is finite. These spaces arise naturally in the theory of Fourier series and it was realized at an early stage of their development that many properties of H^p functions belong to real-variable theory. For, each H^p function can be written as the Poisson integral of an L^p function on the boundary, or, for $p=1$, of a measure. This allows some results to be deduced without using the theory of analytic functions, and it turned out that the portion of the theory susceptible to this treatment is considerable.

In this approach to H^p space theory the basic vehicle for the analysis of the functions in the Hardy spaces is the disc algebra $\mathbb{A}$ on the circle. The space H^p is identified as the closure of $\mathbb{A}$ in the L^p space of $\mathbb{T}$.

In a series of papers, the Hardy space theory on the circle was generalized by Helson and Lowdenslager to the context of certain abelian compact groups. The great generality of their arguments was soon recognized and the theory was successively generalized, in the context of function algebras, first to Dirichlet algebras and then to logmodular algebras. It was ultimately realized that the theory could be extended to the situation where one had a unique representing measure for a character of a function algebra.

Suppose that A is a function algebra on a compact Hausdorff space Ω and that m is the unique representing measure for some character of A (if, for example, A is a logmodular algebra, then, as observed above, every character admits a unique representing measure). To avoid trivialities, we shall also assume that m is not a point mass. For $1\leq p\leq\infty$, we denote by L^p the Lebesgue space $L^p(\Omega, m)$ and for p finite we denote by H^p the norm closure of A in L^p. We signify by H^∞ the weak* closure of A in $L^\infty = L^{1^*}$. The spaces H^p are Banach spaces and, in particular, H^2 is a Hilbert space and H^∞ is a Banach algebra.

Let $\tilde{\Omega}$ be the spectrum of the algebra L^∞. Then the Gelfand representation induces an isometric isomorphism of L^∞ onto $C(\tilde{\Omega})$. Moreover, the image $\tilde{A}$ of the algebra H^∞ under this representation is a function algebra on $\tilde{\Omega}$ and it turns out that $\tilde{A}$ is logmodular. In fact, the following is true: If φ is a real-valued function in L^∞, then there exists an invertible element ψ of H^∞ such that $\varphi = \log|\psi|$. Since a Dirichlet algebra necessarily contains all continuous idempotent functions, since L^∞ is the closed linear span of its idempotent elements, and finally since $H^\infty \neq L^\infty$, it follows that $\tilde{A}$ is an example of a logmodular algebra which is not Dirichlet.

In the case of the disc algebra on the circle, normalized Lebesgue measure is the unique representing measure for a character of A. The corresponding algebra H^∞ is therefore logmodular. This enables one to embed analytic structure into the Gleason parts of its spectrum X. Actually, X is a very complicated space. One can naturally embed the open unit disc into X. A deep result concerning X is the well-known corona theorem of Carleson, which asserts that the disc is dense in X. This can be reformulated as follows: If $f_1,\ldots,f_n$ are bounded analytic functions on the open unit disc such that $\sum_{k=1}^n|f_k|$ is bounded away from zero, then there exist bounded analytic functions $g_1,\ldots,g_n$ such that $f_1g_1+\cdots+f_ng_n=1$.

Let us return to the general situation. We give some examples of the analytic-type properties enjoyed by the elements of H^1. Firstly, the only real-valued elements of H^1 are the real constants. In the case of the circle, this is easily seen, using the fact that an integrable function whose Fourier transform vanishes must itself vanish almost everywhere. In the general situation the proof requires more work.

Analytic functions cannot vanish on "big" sets without vanishing identically. A similar result holds for H^1 space functions. This is our second example of analyticity, and it is a consequence of Jensen's inequality: If f is an element of H^1 such that $\int f\,dm\neq 0$, then $\log|f|$ is integrable with respect to m and

$$\log\left|\int f\,dm\right| \leq \int \log|f|\,dm.$$

Hence, f cannot vanish on a set of positive measure. In the case of the circle, one can strengthen this to assert that if f is a non-zero element of H^1, then f cannot vanish on a set of positive measure, a result known as the little F. and M. Riesz theorem, one of the nicest results of the theory.

The little F. and M. Riesz theorem is an example of a classical result that does not carry over to the general situation. There exists an example of a "system" Ω, A and m, a non-zero function $f \in H^1$ and a set E of positive measure (with respect to m) such that f vanishes on E. Moreover, one may even take f to be an element of A and E to be an open set [19]. This counterexample has consequences in the theory of Toeplitz operators. Incidentally, "systems" Ω, A and m where this kind of behaviour occurs do not have to be pathological in any way.

In passing, one should perhaps observe that the surprising thing is not that some of the classical H^p space theory does not extend to the general situation, but that so much of it does.

Since we have referred to a little F. and M. Riesz theorem, by implication there should be a big F. and M. Riesz theorem, and, of course, there is. In the case of the circle, this well known result asserts that if a Borel complex measure on $\mathbf{T}$ has Fourier–Stieltjes transform supported in $\mathbf{Z}^+$, then it is absolutely continuous with respect to Lebesgue measure on $\mathbf{T}$. This theorem *does* extend to the general situation, although in substantially modified form [3].

In the general context there is also a version of the celebrated invariant subspace theorem of Beurling. A closed vector subspace M of H^2 is *invariant* if $\varphi M \subseteq M$ for all $\varphi \in A$. The generalized Beurling theorem asserts that if M is an invariant space and if there exists $f \in M$ such that $\int f\,dm \neq 0$, then there exists a function $\varphi \in H^\infty$ such that $|\varphi| = 1$ almost everywhere with respect to m and $M = \varphi H^2$ (such functions φ are called *inner* functions).

We now give a very brief sketch of how Wermer's embedding theorem is established. We shall leave out all the technical details of the proof, but nevertheless, the sketch should give a reasonable outline idea of the argument.

Let A be a Dirichlet algebra on Ω. If P is a Gleason part of the spectrum of A that is not a singleton, and m is the unique representing measure for a character τ in P, then one can show

that there is an inner function Z in H^∞ such that

$$ZH^2 = \{f \in H^2 \mid \int f\,dm = 0\}.$$

If Δ denotes the open unit disc and $s \in \Delta$, one can define τ_s in the spectrum of A by $\tau_s(\varphi) = \int \varphi(1 - s\bar{Z})^{-1}\,dm$. Observe that

$$\tau_s(\varphi) = \sum_{n=0}^{\infty}\Big(\int \varphi\bar{Z}^n\,dm\Big)s^n,$$

so the function

$$\Delta \to \mathbf{C}, \quad s \mapsto \tau_s(\varphi),$$

is analytic. Using Schwarz's lemma one can show that

$$\|\tau_s - \tau\| \leq 2|s| < 2$$

for all $s \in \Delta$; therefore, $\tau_s \in P$. One can now show that the map

$$\theta\colon \Delta \to P, \quad s \mapsto \tau_s$$

is a homeomorphism of Δ onto P, where P has the metric topology. For all $\varphi \in A$, the composition $\hat{\varphi} \circ \theta$ is analytic, as $\hat{\varphi}\theta(s) = \tau_s(\varphi)$. Thus we have embedded analytic structure into the spectrum of A. For full details of this construction, see [3].

5. Applications to operator theory

The class of Toeplitz operators is an exceptional class of operators, for it is one of the few large classes of operators about which we have detailed knowledge. Toeplitz operators are related to multiplication operators, but their structure and properties are much more difficult to analyse.

If $\varphi \in L^\infty(\mathbf{T})$, the multiplication operator corresponding to φ is the operator M_φ on $L^2(\mathbf{T})$ defined by $M_\varphi(f) = \varphi f$. The compression of this operator to the Hardy space H^2 is the corresponding Toeplitz operator. Explicitly, if P is the projection of L^2 onto

H^2, then the Toeplitz operator T_φ is defined by $T_\varphi(f) = P(\varphi f)$. The fact that the map, $\varphi \mapsto M_\varphi$, is an algebra homomorphism helps enormously in the analysis of the operators M_φ. For Toeplitz operators the corresponding map, $\varphi \mapsto T_\varphi$, can be easily seen to be linear and to preserve involutions (that is, $T_{\bar\varphi} = T_\varphi^*$), but it is not multiplicative, a fact that makes the analysis of Toeplitz operators very different from that of multiplication operators.

Before proceeding to discuss Toeplitz operators in more detail, let us say a few words about the significance of this class of operators. As indicated above, they are important in operator theory, because they provide a highly non-trivial class of operators accessible to detailed analysis. There are beautiful connections with function theory, specifically H^p space theory. Amongst other applications, there are applications to the analysis of boundary-value problems, to information theory and to time-series analysis in statistics. For more information on applications, see [2].

Suppose now that Ω is a compact Hausdorff space, A is a function algebra on Ω, and m is the unique representing measure for a character of A. As before, we denote the corresponding Lebesgue and Hardy spaces by L^p and H^p, respectively. Given $\varphi \in L^\infty$, one can define the Toeplitz operator T_φ in the same way as in the case of the circle. It is obvious that $\|T_\varphi\| \leq \|\varphi\|_\infty$, and, in fact, equality holds, but this is very non-obvious. One can derive this from a stronger result, a spectral inclusion result which says that the spectrum of T_φ contains the spectrum of φ. (The spectrum is an important invariant. For an element a of a unital Banach algebra, its *spectrum* is the set of all scalars λ such that $a - \lambda$ is not invertible.) This spectral inclusion result is due to Hartman and Wintner [9] in the case of the circle and to the author [17] in the general case, where the proof is quite different to that of the classical case. The proof uses the fact that every real-valued function in L^∞ is the logarithm of the modulus of an element of H^∞.

One of the deepest results concerning Toeplitz operators on the circle is the theorem of Widom [6] which asserts that they have connected spectra. In the general situation the author has shown connectedness of the spectrum still persists for two important subclasses of Toeplitz operators namely, for Hermitian Toeplitz operators T_φ (where $\bar\varphi = \varphi$) and for analytic Toeplitz operators (where $\varphi \in H^\infty$), see [17].

Let us illustrate the interplay of operator theory and function theory by considering a special simple result: A non-scalar Hermitian Toeplitz operator on the circle has no eigenvalues. The proof is easily reduced to showing that zero is not an eigenvalue. Suppose then φ is a real-valued element of L^∞ and that f is an element of H^2 such that $T_\varphi(f) = 0$. Then $P(\varphi f) = 0$, so $\varphi \bar f$ belongs to H^2. Hence, $\varphi \bar f f$ is a real-valued element of H^1, and therefore it is almost everywhere equal to a constant, c say. However, $c = \int c\,dm = \int \varphi \bar f f\,dm = \langle T_\varphi(f), f\rangle = 0$. Thus, $\varphi \bar f f = 0$ a.e. Now the assumption that T_φ is non-scalar assures us that on some set of positive measure, φ does not vanish. Hence, f *does* vanish on a set of positive measure. Therefore, by the F. and M. Riesz theorem, $f = 0$ a.e.

We mentioned earlier that the little F. and M. Riesz theorem does not hold in the general situation. One way that this is reflected in the theory of Toeplitz operators is that non-scalar Hermitian Toeplitz operators on generalized Hardy spaces may have eigenvalues. Even here, however, a striking result is true. Non-zero eigenspaces must be infinite dimensional. For more details, see [17].

Limitations of space have allowed us to give no more than an inkling of the scope of the theory of Toeplitz operators and its profound connections with function theory. Some of the classical theory is covered in [6]. Both the classical and generalized theories are now so extensive that the bibliography that follows can indicate only a few of the many possibilities for further reading for the interested reader.

Toeplitz theory also has connections with the theory of C*-algebras and with K-theory. This aspect of the subject involves index theory, one of the most active areas of modern operator theory. Some references for this are: [4], [6], [12], [13], [16], [18] and [21].

As an introduction to function algebras, Browder's book [3] probably cannot be surpassed. For a more detailed treatment of the subject see the books of Gamelin [7] and Leibowitz [11], each of which contains extensive bibliographies. Two other books which may be consulted are [19] and [20]. Two journal articles that are particularly readable are [1] and [10].

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G. J. Murphy
Department of Mathematics,
University College,
Cork.

THE JOYS OF COLLECTING MATHEMATICAL BOOKS

Rod Gow

I would like to describe an engrossing hobby that I took up about two years ago, namely, the collecting of mathematical books of some antiquity or historical value. This activity has proved to be an impetus for discovering more about the historical development of mathematics, both at the research and teaching level. I hope that readers will find some aspects of the economics and mechanics of collecting specialized books to be of interest.

My introduction to the world of antiquarian books dates from the death in 1990 of my father–in–law Con Gillman, who was a graduate in physics of UCC and possessor of an M.Sc. in statistics. He left a large number of books in the study and attic of his house. These books posed a problem of dispersal, as they occupied much space and would eventually have to be moved. Examination of these books revealed titles in philosophy, poetry, mathematics, statistics, physics, meteorology and engineering. While I was not particularly enthusiastic about many of the books, I was impressed by a group of physical and mathematical books that had probably been collected in the 1930's or 40's. These included works by such authors as Rutherford, J. J. Thomson, Dirac, Fermi, W. Thomson (Lord Kelvin) with whom I was more or less familiar. Other authors included W. Stanley Jevons, John H. Pratt, Isaac Todhunter and Joseph Wolstenholme, names which then meant nothing to me. My attempts to find out more about these authors and their work have led me into the fascinating world of the history of science and scientific literature and ultimately to that of book collecting.

An obvious source for enquiring about British and Irish scientists or academics of earlier times is the *Dictionary of National*

Biography or DNB. The DNB has sometimes provided the only information I have been able to find on authors whose work I have encountered. For instance, I read that Wolstenholme (1829–91) was a clergyman, a fellow of Christ's College, Cambridge and professor of mathematics at the Royal Indian Engineering College in London. He is best known for his *Mathematical Problems*, published in three editions (1867, 1878 and 1891) by Macmillan and Co. He apparently did much work as an examiner for the mathematical tripos in Cambridge and as a consequence built up a large store of problems, which formed the basis of his book. The 1878 version has 2815 problems, occupying 480 pages. Unfortunately, most of these problems seem to be unsuitable for present day students (there are 936 problems on conic sections but only 26 on probability). Another useful source of information is the *Encyclopedia Britannica*. The earlier editions of *Britannica* often had detailed biographical and expository articles, which were subsequently edited and shortened in later editions, thereby losing much of their charm. Copies of various editions (but not the most recent or earliest ones) of *Britannica* can often be bought at auction or from dealers for around £100-150 and are a great asset to the bibliophile or amateur of scientific history.

A more detailed source is the *Dictionary of Scientific Biography* or DSB, edited by C. Gillispie and published (1970–76) by Charles Scribner's Sons. This consists of 14 volumes, together with two supplements, and contains biographical articles on people connected with science and mathematics from Greek times up to the present century. Many of these articles are full and authoritative, being written by named experts in the relevant field, although occasionally, some seem rather short and disappointing. Many of the minor scientists, about whose work or books I may wish to have more detail, probably have not merited attention in the DSB, although they often feature in the DNB. While a personal copy of the DSB would be invaluable to any enthusiast of the history of science, second–hand sets are not so easily obtained and would probably cost at least £700. More recently (1991), the articles from the DSB relating to mathematicians have been published in four separate volumes.

Having found information about the authors of the books I had acquired, I then wondered if any of the books was at all rare or valuable. The novice, when confronted by antiquarian books, will usually have little idea of what might be a reasonable price to pay for any particular item. In some respects, this will always remain a problem, as prices can fluctuate wildly, depending on who is selling. One may come across bargains, where the seller has little idea of the value attached to an item by an enthusiast. Equally, absurdly high prices may be demanded by dealers who have bought relatively cheaply and intend to sell on expensively. The problems of buying and selling in a very small market are delicate and probably do not conform to the more normal economics of production and retail of essential commodities.

There are various ways to obtain some picture of how much antiquarian or rare books cost. One is to visit as many second–hand book shops as possible and see what prices are demanded for particular items. However, if your interest is in scientific literature, there will not be many such shops that have more than a few isolated examples to examine. In Dublin, there are several second–hand and antiquarian book shops and dealers, but most specialize, not surprisingly, in Irish literature, history, topography and travel. Nonetheless, occasionally something of scientific merit may creep in and a certain amount of sustained but generally fruitless searching will locate a worthwhile item. Another possibility is to find the names of specialized dealers who conduct their business by mail and write to them for catalogues. I have made use of two such dealers, who advertise their facilities in mathematical magazines. Their catalogues are very instructive and provide quick access to information about prices and rarity. I noted a copy of Hamilton's *Lectures on Quaternions* of 1853 on sale in one catalogue at £180, which struck me as reasonable value (doubtless it was sold almost instantly). Often, dealers whose speciality is not in science may have one or two scientific items and I will usually try to browse through any catalogue that comes my way. It is clear that any pre–19th century mathematical or scientific book is likely to cost at least £30 if it is not badly damaged and often more than £100 if the author is famous or the condition is good.

Pre–18th century scientific books are now very uncommon and unlikely to be obtained, except from top–class dealers, who often charge top–class prices. On the whole, for the purchaser of limited means, pre–18th century scientific material is largely out of the question.

Instead of buying through a dealer, one may decide to try buying at auction. In London, Christies and Bloomsbury Book Auctions, among others, have annual auctions of scientific books. I have never attended any such auction, but many dealers obtain their stock by buying at these specialized auctions. A valuable guide in this respect is *Book Auction Records* or BAR, published by Dawson, which has appeared annually for many decades. The BAR provides information on sales in auction houses throughout the world, the condition for inclusion of any sale being that the item sold must have fetched at least £70. I enjoy browsing through the copies of BAR and have noticed various sales of mathematical note. For example, a copy of Boole's *An Investigation of the Laws of Thought* sold in 1991 for £3,900 at an auction conducted by Dominic Winter's of Swindon. This copy once belonged to W. S. Jevons and had a letter of 1868 from Augustus de Morgan to Jevons tipped into it. Boole's book was published by Macmillan and Co. in 1854 and was still described in Macmillan's 1879 catalogue as available for 14s. While the price paid above was obviously enhanced by the association with Jevons, any copy is likely to cost hundreds of pounds nowadays. It is sometimes possible to identify an item bought at auction with a similar item in a dealer's catalogue and thus deduce the dealer's mark up. One must expect a mark up of at least 50% but one of 200% or 300% is not unknown.

In Ireland, the main book auctions are those conducted by Mealy's of Castlecomer. These occur twice or more a year, in the Montrose Hotel in Dublin or in Castlecomer. Mealy's auctions usually have one or two lots of mathematical interest but they cater more for the Irish literary and historical tastes that I described above. In December 1991, before the start of my interest, Mealy's Dublin sale had three mathematical lots: a first edition of Thomas Simpson's *Treatise of Algebra* of 1745 (the

Simpson of Simpson's rule), a second edition of Robert Simson's *Sectionum Conicarum Libri V* of 1750 (Simson was professor of mathematics in Glasgow and the editor of a version of Euclid that was much followed and imitated for many years) and a third edition of Newton's *Principia* of 1726. Copies of the 1726 *Principia* are not nearly as rare as those of the first edition of 1687. Perusal of the BAR suggests that £400-1000 might be the price to pay for the 1726 *Principia*, depending on condition, historical associations and so on. I wonder who bought the three books above and what they paid.

I have related how my interest in antiquarian scientific and mathematical books was awakened and how I tried to learn more about their history and prices. Enthused by this initial impetus, I decided in 1992 that, using my father–in–law's books as a nucleus, I would try to extend his collection, specializing in mathematical work, with a subsidiary interest in other scientific subjects. So, collecting mathematical books has become a new hobby. There are probably not many people in Ireland with similar interests in mathematical books, so that competition in the market is small but, on the reverse side, there are not many outlets for the material I seek. I might add that, as far as I can see from my investigations, the most valuable of my father–in–law's books is probably *Histoire des mathématiques* by Jean Montucla (1725-99). This consists of two quarto volumes published in Paris in 1758. It is considered to be one of the best early histories of mathematics.

Let me describe some of the books that I have acquired. There is a copy of the second issue of Legendre's *Elements of Geometry* of 1824, edited by David Brewster. (Virtually no copies of the first issue of 1822 are known.) This is an English translation of the eleventh edition of Legendre's French original, which first appeared in 1794. The translation was made by Thomas Carlyle, who is better known as an essayist and historian. Carlyle included an introduction 'On Proportion', written by himself. He was paid £50 for the translation. According to an article by J. H. Webb in the *Mathematical Gazette* for June 1974, Carlyle eventually tired of the translating work and persuaded his brother to take over. In Legendre's French text there is a definition of a straight line by 'La ligne droite est le plus court chemin d'un point à un autre'. This was mistranslated by Carlyle as 'A straight line is the shortest distance from one point to another' (the mistranslation occurring in the word 'chemin', which means 'path' rather than 'distance').

I was particularly pleased to buy *A History of the Mathematical Theory of Probability from the Time of Pascal to that of Laplace* by Isaac Todhunter, published in 1865 by Macmillan and sold at a price of 18s. Even today, historians of probability and statistics refer to Todhunter's text for its detailed description of the development of the first 150 years of the theory. The work is described as *very scarce* in the catalogue *Bibliotheca Chemico–Mathematica* of 1921 and copies of it were offered for sale at £2 2s, £2 7s 6d and £2 15s. Another classical text that I obtained on probability theory is *Choice and Chance* by W. A. Whitworth. This appeared in five editions between 1867 and 1901. The edition I bought was the fourth, published in 1886. It contains 640 exercises on elementary probability theory and combinatorics and is still useful to anyone who has to devise problems for a finite mathematics course. Incidentally, Whitworth was vicar of All Saints', Margaret Street, London, a well–known centre of High Church Anglicanism.

I mentioned the *Bibliotheca Chemico–Mathematica* above and I will take the opportunity to describe the virtues of this publication. It is an historical catalogue of scientific books that were available for sale at Henry Sotheran and Co. of London. (Sotheran's is still one of the major dealers in antiquarian books in London.) The work is quite extensive and a valuable source for those wishing to research scientific literature. It is now a collector's item and well worth owning by the enthusiast. A look at prices is instructive. A first edition of *Principia* was available for 18 guineas and was described as *excessively scarce*. The going rate at auction for such a first edition seems to be £20,000-30,000 for a good copy nowadays. It is not known exactly how many copies of the first edition of *Principia* were printed, although estimates of between 250 and 400 copies have been made. Leather bound copies of the book retailed for 9s initially but apparently the book had become scarce very quickly and people were paying much more for second–hand copies. In 1691, it is recorded that a copy was bought for about 2 guineas. It should be borne in mind that a good husbandman could be hired for between £3 and £4 a year (information from the *Mathematical Gazette* for December 1948).

Many of the books in *Bibliotheca Chemico–Mathematica* now cost between 500 and 1000 times what they cost 70 years ago. This is particularly true of books whose authors have become famous (like Babbage or Boole) or books produced in small print runs. (A sum invested at 10% interest compounded annually would increase 1000 fold in about 72 years.) On the other hand, some text books have really declined in value over the years when inflation is taken into account. For example, the first edition of Todhunter's *Analytical Statics* was published by Macmillan in 1853 and sold at 10s 6d, which seems expensive for the time. The fifth edition of the book was still on sale in 1890 at the same price. Today, this book would probably cost no more than £15, so that it has not held its value over 140 years. Boole's *Laws of Thought* was available from Sotheran's for £1 15s and it was described as *very scarce*. Group theorists who know the impact made in the last century by Camille Jordan's *Traité des substitutions*, published in 1870, may like to know that this work was available from Sotheran's for £3 7s 6d and described as *very scarce*. I have not heard of an original copy for sale in recent times.

In conclusion, I would like to make a small advertisement. I am interested in buying books, papers or magazines of a scientific or mathematical nature, preferably pre–20th century. If you have any such items that you wish to dispose of, you may think of contacting me at the address below.

Department of Mathematics
University College
Belfield
Dublin 4
email: rodgow@irlearn.ucd.ie

POLYNOMIALS AND SERIES IN BANACH SPACES*

Manuel González† and Joaquín M. Gutiérrez‡

Abstract: We show that homogeneous polynomials acting on Banach spaces preserve weakly unconditionally Cauchy (w.u.C.) series and unconditionally converging (u.c.) series. This fact allows to define the class of unconditionally converging polynomials as those taking w.u.C. series into u.c. series. It includes most of the classes of polynomials previously considered in the literature. Then we study several "polynomial properties" of Banach spaces, defined by relations of inclusion between classes of polynomials. In our main result we show that a Banach space E has the polynomial property (V) if and only if for all $k \in \mathbf{N}$ the space of homogeneous scalar polynomials $\mathcal{P}(^kE)$ is reflexive; hence, its dual space E^*, like the dual of Tsirelson's space, is reflexive and contains no copies of ℓ_p.

Throughout the paper, E and F will be real or complex Banach spaces, B_E the unit ball of E and E^* its dual space. We will write $\mathbf{K}$ for the scalar field, which will be always $\mathbf{R}$ or $\mathbf{C}$, the real or the complex field, and $\mathbf{N}$ for the natural numbers. Moreover, $\mathcal{P}(E,F)$ will stand for the space of all (continuous) polynomials from E into F. Any polynomial $P \in \mathcal{P}(E,F)$ can be written as a sum of homogeneous polynomials: $P = \sum_{k=0}^{n} P_k$, with $P_k \in \mathcal{P}(^kE,F)$, the space of all k-homogeneous polynomials from E into F.

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We will only give here the main results and sketches of some of the proofs. A complete exposition, including detailed proofs, can be found in [5].

1. Preservation of series by polynomials

Recall that a series $\sum_{i=1}^{\infty} x_i$ in a Banach space E is *weakly unconditionally Cauchy* (in short, *w.u.C.*) if for every $f \in E^*$ we have $\sum_{i=1}^{\infty} | f(x_i) | < \infty$; equivalently, if

$$\sup_{|\epsilon_i| \leq 1} \left\| \sum_{i=1}^{\infty} \epsilon_i x_i \right\| < \infty.$$

The series $\sum_{i=1}^{\infty} x_i$ is *unconditionally convergent* (in short, *u.c.*) if any subseries is norm-convergent; equivalently, if

$$\lim_{n \to \infty} \sup_{|\epsilon_i| \leq 1} \left\| \sum_{i=n}^{\infty} \epsilon_i x_i \right\| = 0.$$

The series $\sum_{i=1}^{\infty} x_i$ is *absolutely convergent* if $\sum_{i=1}^{\infty} \|x_i\| < \infty$.

Clearly, absolutely convergent series are u.c.; however, by the Dvoretsky-Rogers theorem [4], any infinite-dimensional Banach space contains an u.c. series which is not absolutely convergent. For example, if e_n denotes the unit vector basis of ℓ_2, then the series $\sum_{n=1}^{\infty} e_n/n$ is u.c., but not absolutely convergent.

Also, any u.c. series is w.u.C., and the prototype example of w.u.C. series which is not u.c. is given by $\sum_{i=1}^{\infty} e_i$, where $\{e_i\}$ denotes the unit vector basis of the space c_0. In fact, for any w.u.C. series $\sum_{i=1}^{\infty} x_i$ which is not u.c. there exist natural numbers $m_1 < n_1 < \cdots < m_k < n_k < \cdots$ such that the sequence of blocks

$$y_k := x_{m_k} + \cdots + x_{n_k}$$

is equivalent to the unit vector basis of c_0. (See [4]).

In this section we give first an estimation of the unconditional norm of the image of a finite sequence by a homogeneous polynomial, from which we derive the preservation of w.u.C. series and u.c. series under the action of polynomials. Then we define the class of unconditionally converging polynomials, and compare it with other classes of polynomials that have appeared in the literature.

Lemma 1. *Given $k \in \mathbb{N}$ there exists $C_k > 0$ such that for any $P \in \mathcal{P}(^kE, F)$ and $x_1, ..., x_n \in E$ we have*

$$\sup_{|\epsilon_j| \leq 1} \left\| \sum_{j=1}^{n} \epsilon_j P x_j \right\| \leq C_k \sup_{|\nu_j| \leq 1} \left\| P \left(\sum_{j=1}^{n} \nu_j x_j \right) \right\|.$$

We can take $C_k = 1$ in the complex case, and $C_k = (2k)^k/k!$ in the real case.

The proof of the case in which E and F are complex spaces relies on the properties of the generalized Rademacher functions $s_n(t)$, $n \in \mathbb{N}$, introduced in [3], which are step functions on the interval $[0,1]$ verifying [3] for any choice of integers $i_1, \ldots, i_k$; $k \geq 2$,

$$\int_0^1 s_{i_1}(t) \ldots s_{i_k}(t) dt = \begin{cases} 1 & \text{if } i_1 = \cdots = i_k; \\ 0 & \text{otherwise.} \end{cases}$$

In the case of real spaces, the proof is obtained using the complexifications of the spaces, and the polarization identities relating a homogeneous polynomial and its associated symmetric multilinear map.

Using Lemma 1, it is not difficult to prove the following

Theorem 2. *Any polynomial $P \in \mathcal{P}(E, F)$ takes w.u.C. (u.c.) series into w.u.C. (u.c.) series.*

This result suggests introducing the following class of polynomials.

Definition 3. A polynomial $P \in \mathcal{P}(E, F)$ is said to be *unconditionally converging* if it takes w.u.C. series into u.c. series.

We shall denote by $\mathcal{P}_{uc}(^kE, F)$ the class of all k-homogeneous unconditionally converging polynomials from E to F.

Observe that in the case of E or F containing no copies of c_0, any w.u.C. series in that space is u.c. [4]; hence $\mathcal{P}(^kE, F) = \mathcal{P}_{uc}(^kE, F)$. Moreover, we can characterize unconditionally converging polynomials in terms of the action on sequences equivalent to the unit vector basis $\{e_n\}$ of c_0.

Lemma 4. *For any $P \in \mathcal{P}(^kE, F)$ which is not unconditionally converging there exists an isomorphism $i : c_0 \to E$ such that $\{(P \circ i)e_n\}$ is equivalent to $\{e_n\}$.*

The proof uses the Bessaga-Pełczyński principle in order to select a basic sequence from certain blocks of a suitable w.u.C. series $\sum_{i=1}^{\infty} x_i$ such that $\sum_{i=1}^{\infty} Px_i$ is not u.c., and then applies Lemma 1.

Next we describe the relation between the class $\mathcal{P}_{uc}$ and other classes of polynomials considered in the literature.

Recall that $P \in \mathcal{P}(^kE, F)$ is *weakly compact*, denoted by $P \in \mathcal{P}_{wco}(^kE, F)$, if it takes bounded subsets into relatively weakly compact subsets, and P is *completely continuous*, denoted by $P \in \mathcal{P}_{cc}(^kE, F)$, if it takes weakly Cauchy sequences into norm convergent sequences. These classes were considered in [10] and [11].

Moreover, we shall consider the class $\mathcal{P}_{cc0}(^kE, F)$ of *completely continuous at 0* polynomials, formed by those $P \in \mathcal{P}(^kE, F)$ taking weakly null sequences into norm null sequences. Clearly $\mathcal{P}_{cc}(^kE, F) \subset \mathcal{P}_{cc0}(^kE, F)$, but in general (see Proposition 14) the containment is strict for $k > 1$ and E failing the Schur property.

Recall that $A \subset E$ is said to be a *Rosenthal set* if any sequence $(x_n) \subset A$ has a weakly Cauchy subsequence. In contrast with the case of linear operators, a polynomial taking Rosenthal sets into relatively compact subsets need not take weakly null sequences into norm null sequences, as it is shown by the scalar polynomial

$$P : (x_n) \in \ell_2 \longrightarrow \sum_{n=1}^{\infty} x_n^2 \in \mathbf{R}.$$

The converse implication also fails, since for the polynomial

$$Q : (x_n) \in \ell_2 \longrightarrow \left(\sum_{k=1}^{\infty} \frac{x_k}{k} \right) (x_n) \in \ell_2$$

we have that $Q(e_1 + e_n) = (1 + 1/n)(e_1 + e_n)$ has no convergent subsequences, although Q takes weakly null sequences into norm null sequences, because of the factor $(\sum_{k=1}^{\infty} x_k/k)$.

Finally, recall that $A \subset E$ is said to be a *Dunford-Pettis set* [2] if for any weakly null sequence $(f_n) \subset E^*$ we have $\lim_n \sup_{x \in A} |f_n(x)| = 0$. Using this class of subsets, we will say as in [5] that a polynomial $P \in \mathcal{P}(^kE, F)$ belongs to $\mathcal{P}_{wd}$ if and only if its restriction to any Dunford-Pettis subset of E, endowed with the inherited weak topology, is continuous.

Proposition 5. *A polynomial $P \in \mathcal{P}(^kE, F)$ belongs to $\mathcal{P}_{uc}$ in the following cases:*
(a) $P \in \mathcal{P}_{cc0}$.
(b) P takes Rosenthal subsets of E into relatively compact subsets of F.
(c) $P \in \mathcal{P}_{wd}$.
(d) $P \in \mathcal{P}_{wco}$.

The result in the cases (a) and (b) is an immediate consequence of Lemma 4, since the unit vector basis of c_0 is a weakly null sequence which forms a non relatively compact set.

Case (c) follows from Lemma 4 also, since given $P \in \mathcal{P}(^kE, F)\backslash\mathcal{P}_{uc}$, and an isomorphism $i : c_0 \to E$ such that $P \circ i \notin \mathcal{P}_{uc}(^kc_0, F)$, we have that $\{ie_n\}$ is a Dunford-Pettis set of E on which P is not weakly continuous; hence $P \notin \mathcal{P}_{wd}$.

Finally, we have $\mathcal{P}_{wco}(^kE, F) \subseteq \mathcal{P}_{wd}(^kE, F)$ (see [5]); hence (d) follows from (c).

2. Polynomial properties of Banach spaces

Pełczyński [10] introduced Banach spaces with the *polynomial Dunford-Pettis property* as the spaces E such that (with our notation) $\mathcal{P}_{wco}(^kE, F) \subseteq \mathcal{P}_{cc}(^kE, F)$ for any $k \in \mathbf{N}$ and F, and raised the question whether or not the polynomial Dunford-Pettis property coincides with the usual *Dunford-Pettis property*, which admits the same definition in terms of linear operators ($k = 1$). Ryan gave an affirmative answer in [11]. Moreover, Pełczyński [9] introduced Banach spaces with *property (V)* as the spaces E such that unconditionally converging operators from E into any Banach space are weakly compact.

In this section, by means of the class $\mathcal{P}_{uc}$ of unconditionally converging polynomials, we introduce and study the polynomial property (V) and other polynomial versions of properties of Banach spaces viz: the Dieudonné property, the Schur property, and property (V^*). We show that in contrast with the case of the Dunford-Pettis property, property (V) is very different from the polynomial property (V), since the prototype of space with this property is Tsirelson's space T^*. For the other polynomial properties, we show that sometimes the polynomial and the linear properties coincide, and sometimes not, with a general tendency of the polynomial property to imply the absence of copies of ℓ_1 in the space. Moreover, we obtain additional results relating $\mathcal{P}_{uc}$ and other classes of polynomials.

Definition 6. A Banach space E has the *polynomial property (V)* if for every k and F we have $\mathcal{P}_{uc}({}^kE,F) \subseteq \mathcal{P}_{wco}({}^kE,F)$.

It was shown in [9] that $C(K)$ spaces enjoy property (V). The next Lemma shows that this is not the case for the polynomial property.

Lemma 7. *If $\mathcal{P}_{uc}({}^kE,E) \subseteq \mathcal{P}_{wco}({}^kE,E)$ for some $k > 1$, then E contains no copies of c_0.*

It has been shown that a Banach space E such that $\mathcal{P}({}^kE,\mathbf{K}) \equiv \mathcal{P}({}^kE)$ is reflexive for every $k \in \mathbf{N}$ has many of the properties of Tsirelson's space T^* [14]. In fact, E must be reflexive, and the dual space E^* cannot contain copies of ℓ_p $(1 < p < \infty)$. Note also that $\mathcal{P}({}^kT^*)$ is reflexive for every $k \in \mathbf{N}$ [1]. Next we present a characterization of the spaces E such that $\mathcal{P}({}^kE)$ is reflexive for some $k > 1$ in terms of the class $\mathcal{P}_{uc}$ of polynomials.

Given $P \in \mathcal{P}({}^kE,F)$, we consider the associated conjugate operator defined by

$$P^* : f \in F^* \longrightarrow f \circ P \in \mathcal{P}({}^kE).$$

Moreover, we need the fact that for every Banach space E, the space $\Delta_\pi^k E$, defined as the closed span of $\{x \otimes \cdots \otimes x : x \in E\}$ in the projective tensor product $\hat{\otimes}_\pi^k E$, is a predual of the space of scalar polynomials $\mathcal{P}({}^kE)$ [12].

Theorem 8. *Given $k > 1$, we have that the space $\mathcal{P}({}^kE)$ is reflexive if and only if $\mathcal{P}_{uc}({}^kE,F) \subseteq \mathcal{P}_{wco}({}^kE,F)$ for any F . In particular, E has the polynomial property (V) if and only if $\mathcal{P}({}^kE)$ is reflexive for every $k \in \mathbf{N}$.*

For the direct result it is enough to note that $P \in \mathcal{P}_{wco}$ if and only if the operator P^* is weakly compact [13].

For the converse, we derive from Lemma 7 that E contains no copies of c_0; hence $\mathcal{P}({}^kE,F) = \mathcal{P}_{uc}({}^kE,F) = \mathcal{P}_{wco}({}^kE,F)$ for any k and F, and then we observe that there exists a natural isomorphism between the space of polynomials $\mathcal{P}({}^kE,F)$ and the space of operators $L(\Delta_\pi^k E,F)$ which takes the weakly compact polynomials onto the weakly compact operators [12].

Extending the definition for operators, we shall say that $P \in \mathcal{P}({}^kE,F)$ is *weakly completely continuous*, denoted by $P \in \mathcal{P}_{wcc}({}^kE,F)$, if it takes weakly Cauchy sequences into weakly convergent sequences.

A Banach space E has the *Dieudonné property* if weakly completely continuous operators from E into any Banach space are weakly compact. Grothendieck [7] introduced this property and proved that $C(K)$ spaces enjoy it. The next result shows that, in general, $C(K)$ spaces fail to satisfy the polynomial Dieudonné property.

Proposition 9. *The following properties are equivalent:*
(a) E contains no copies of ℓ_1.
(b) $\mathcal{P}_{wcc}({}^kE,F) \subseteq \mathcal{P}_{wco}({}^kE,F)$ for any k and F.
(c) $\mathcal{P}_{cc}({}^kE,F) \subseteq \mathcal{P}_{wco}({}^kE,F)$ for any k and F.
(d) $\mathcal{P}_{cc}({}^kE,F) \subseteq \mathcal{P}_{wco}({}^kE,F)$ for some nonreflexive F and some $k > 1$.

Corollary 10. *$\mathcal{P}_{wcc}({}^kE,F) \subseteq \mathcal{P}_{uc}({}^kE,F)$ for any $k \in \mathbf{N}$.*

Remark 11. It follows from Proposition 9 that, for any $k > 1$, there is a polynomial $P \in \mathcal{P}_{cc}({}^k\ell_\infty, c_0)$ which is not weakly compact.

However, any operator from ℓ_∞ into c_0 is weakly compact and thereby completely continuous, since ℓ_∞ has the Dunford-Pettis property.

Then the question arises whether every polynomial from ℓ_∞ into c_0 is completely continuous.

As a complement of Theorem 8 we have the following

Theorem 12. *Given $k > 1$, we have $\mathcal{P}_{cc0}({}^kE, F) \subseteq \mathcal{P}_{wco}({}^kE, F)$ for any F if and only if $\mathcal{P}({}^{k-1}E)$ is reflexive.*

Remark 13. In order to compare Theorems 8 and 12, we observe that for the sequence spaces ℓ_p the space of polynomials $\mathcal{P}({}^k\ell_p)$ is reflexive if and only if $k < p < \infty$.

In fact, it was proved in [8] that for $k < p$, all polynomials in $\mathcal{P}({}^k\ell_p)$ are completely continuous; hence, using a result of [12] (see [1]), we conclude that $\mathcal{P}({}^k\ell_p)$ is reflexive. For $1 < p \leq k$ it is not difficult to show that $\mathcal{P}({}^k\ell_p)$ contains a copy of ℓ_∞.

Recall that a Banach space E has the *Schur property* if weakly convergent sequences in E are norm convergent; equivalently, weakly Cauchy sequences are norm convergent. It is an immediate consequence of the definition that E has the Schur property if and only if $\mathcal{P}({}^kE, F) = \mathcal{P}_{cc}({}^kE, F)$ for any k and F. Next we give some other polynomial characterizations of Schur property.

Proposition 14. *The following properties are equivalent:*
(a) E has the Schur property.
(b) $\mathcal{P}_{uc}({}^kE, F) \subseteq \mathcal{P}_{cc}({}^kE, F)$ for any k and F.
(b') $\mathcal{P}_{cc0}({}^kE, F) \subseteq \mathcal{P}_{cc}({}^kE, F)$ for any k and F.
(c) $\mathcal{P}_{uc}({}^kE, E) \subseteq \mathcal{P}_{cc}({}^kE, E)$ for some $k > 1$.
(c') $\mathcal{P}_{cc0}({}^kE, E) \subseteq \mathcal{P}_{cc}({}^kE, E)$ for some $k > 1$.

Another property defined in terms of series is property (V^*), introduced in [10]. Recall that a subset $A \subset E$ is said to be a *(V^*) set* if for every w.u.C. series $\sum_{n=1}^{\infty} f_n$ in E^* we have

$$\lim_n \sup_{x \in A} |f_n(x)| = 0.$$

A Banach space E has *property* (V^*) if every (V^*) set in E is relatively weakly compact; equivalently, if any operator $T \in L(F, E)$, with unconditionally converging conjugate T^* is weakly compact.

Finally, we shall show that the polynomial version of the last formulation coincides with property (V^*). We shall denote by $\mathcal{P}_{uc*}({}^kF, E)$ the class of all polynomials $P \in \mathcal{P}({}^kF, E)$ such that P^* is unconditionally converging.

Proposition 15. *The following properties are equivalent:*
(a) E has property (V^*).
(b) For any k and any F, we have $\mathcal{P}_{uc*}({}^kF, E) \subseteq \mathcal{P}_{wco}({}^kF, E)$.
(c) For some k, we have $\mathcal{P}_{uc*}({}^k\ell_1, E) \subseteq \mathcal{P}_{wco}({}^k\ell_1, E)$.

In the proof we need the fact that given $P \in \mathcal{P}({}^kF, E)$, the conjugate P^* is unconditionally converging if and only if $P(B_F)$ is a (V^*) set.

Remark 16. Part of the above results can be extended to holomorphic maps on Banach spaces. For example, holomorphic maps preserve u.c. series and w.u.C. series fulfilling natural restrictions, and if we define the *holomorphic property (V)* in the natural way, we can prove that it coincides with the polynomial property (V). For the details we refer to [5].

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Manuel González
Departamento de Matemáticas
Facultad de Ciencias
Universidad de Cantabria
Avda. de Los Castros s.n.
E-39071 Santander (Spain)

Joaquín M. Gutiérrez
Departamento de Matemática Aplicada
ETS de Ingenieros Industriales
Universidad Politécnica de Madrid
José Gutiérrez Abascal 2
E-28006 Madrid (Spain)

Research Announcement

TAYLOR-MONOMIAL EXPANSIONS OF HOLOMORPHIC FUNCTIONS ON FRÉCHET SPACES

Seán Dineen

Let $\lambda := \lambda(A)$ denote a Fréchet nuclear spaces with Köthe matrix A and let $\{E_n\}_n$ denote a sequence of Banach spaces. Let $E := \lambda(\{E_n\}_n) := \{(x_n)_n : x_n \in E_n \text{ and } (\|x_n\|)_n \in \lambda(A)\}$ and endow E with the topology generated by the semi-norms

$$\|(x_n)_n\|_k := \sum_{n=1}^{\infty} a_{n,k}\|x_n\|, \quad k = 1, 2, \ldots$$

E is a Fréchet space and $\{E_n\}_n$ is an unconditional Schauder decomposition of E. Examples of spaces which can be represented in this fashion, include all Banach spaces and all Fréchet nuclear (and some Fréchet-Schwartz) spaces with basis. Let $H(E)$ denote the space of all **C**-valued holomorphic functions on E and for $m \in N^{(N)}$, $m = (m_1, \ldots, m_n, 0 \ldots)$ let

$$P_m(x) = \frac{1}{(2\pi i)^n} \int_{|\lambda_i|=1} \frac{f(\sum_{i=1}^n \lambda_i x_i)}{\lambda_1^{m_1+1} \ldots \lambda_n^{m_n+1}} \, d\lambda_1 \ldots d\lambda_n$$

We have

$$f = \sum_{m \in N^{(N)}} P_m \qquad (*)$$

in the τ_0, τ_w, τ_δ topologies on $H(E)$.

The expansion $(*)$ reduces to the Taylor series expansion in the case of a Banach space (i.e. if $E_1 = E$, $E_n = 0$, $n > 1$) and to the monomial expansion for Fréchet nuclear spaces with a basis (when $\dim(E_n) = 1$ all n).

If $(E_n)_n$ is an unconditional Schauder decomposition for the Fréchet space E then the topology of E is generated by semi-norms satisfying

$$\left\|\sum_{n=1}^{\infty} x_n\right\| = \sup_{|\lambda_n|\le 1}\left\|\sum_{n=1}^{\infty}\lambda_n x_n\right\| \qquad (**)$$

If $(\beta_n)_{n=1}^{\infty}$ is a sequence of real numbers, $\beta_n \ge 1$ all n, let

$$\left\|\sum_{n=1}^{\infty} x_n\right\|_{\beta,j} = \left\|\sum_{n=1}^{j} x_n + \sum_{n=j+1}^{\infty}\beta_n x_n\right\|.$$

If $m \in N^{(N)}$ we let $\mathcal{P}_e(^m E)$ denote the set of all $|m|$-homogeneous polynomials on E which are homogeneous in the even variables i.e. if $m = (m_1, m_2, \ldots, m_n, \ldots)$ then $P \in \mathcal{P}_e(^m E)$ if and only if

(i) $P(\lambda x) = \lambda^{|m|} P(x)$ for all $x \in E$, $\lambda \in \mathbf{C}$.

(ii) $P\left(\lambda x_{2i} + \sum_{n=1}^{\infty} x_n\right) = \lambda^{m_{2i}} P\left(\sum_{n=1}^{\infty} x_n\right)$ for all i, all $x \in E$ and all $\lambda \in \mathbf{C}$.

Our main technical tool is the following proposition.

Proposition. *Let $\{E_n\}_n$ denote an unconditional Schauder decomposition for the Fréchet space E, let F denote a Banach space and let T denote an F-valued linear function on $H(E)$ which is bounded on the locally bounded subsets of $H(E)$. Let $\beta_n \ge 1$ all n, $\beta_{2n-1} = 2$ all n and suppose $\sum_{n=1}^{\infty} x_n \in E$ implies $\sum_{n=1}^{\infty} \beta_n^p x_n \in E$ for all $p > 0$. Let $\|\cdot\|$ denote a continuous semi-norm satisfying $(**)$ and suppose there exists $C > 0$ such that*

$$\|T(P)\| \le C\|P\| \quad \text{for all } P \in \mathcal{P}_e(^m E) \text{ and all } m \in N^{(N)}$$

where $\|P\| = \sup\{|P(x)|;\ \|x\| \le 1\}$. Then, for any $\delta > 1$, there exists $C_1 > 0$ and a positive integer j such that

$$\|T(P)\| \le C_1 \delta^{|m|} \|P\|_{\beta,j}$$

for all $P \in \mathcal{P}_e(^m E)$ and all $m \in N^{(N)}$ where

$$\|P\|_{\beta,j} = \sup\{|P(x)|;\ \|x\|_{\beta,j} \le 1\}.$$

Theorem. *Let $\lambda(A)$ denote a Fréchet-nuclear space with DN and let $\{E_n\}_n$ denote a sequence of Banach spaces each of which admits an unconditional finite dimensional Schauder decomposition. Then $\tau_w = \tau_\delta$ on $\mathcal{H}\big(\lambda(\{E_n\}_n)\big)$.*

A Fréchet nuclear space has DN if and only if it is isomorphic to a subspace of s. The above theorem includes the known cases where E is a Banach space with an unconditional basis and the case where E is a Fréchet-nuclear space with basis and DN. It also includes Fréchet–Schwartz spaces which are not nuclear. By considering complemented subspaces, we find that $\tau_w = \tau_\delta$ on $\mathcal{H}(E)$, where E is any reflexive subspace, with the approximation property, of a Banach space with an unconditional finite dimensional Schauder decomposition. This includes spaces which do not have a finite dimensional Schauder decomposition.

Seán Dineen,
Department of Mathematics,
University College Dublin,
Belfield,
Dublin 4.

Research Announcements

FINITE ELEMENT METHODS FOR SINGULARLY PERTURBED HIGHER ORDER ELLIPTIC TWO-POINT BOUNDARY VALUE PROBLEMS WITH TWO BOUNDARY LAYERS

Guangfu Sun & Martin Stynes

Piecewise polynomial Galerkin finite element methods are constructed on a Shishkin mesh for a class of singularly perturbed two-point boundary value problems of order greater than two. The methods are proved to be convergent, uniformly in the perturbation parameter, in various norms. Some numerical results are presented for a fourth order problem. Full details are in [1].

FINITE ELEMENT METHODS FOR SINGULARLY PERTURBED HIGHER ORDER ELLIPTIC TWO-POINT BOUNDARY VALUE PROBLEMS II: CONVECTION-DIFFUSION TYPE

Guangfu Sun & Martin Stynes

We consider singularly perturbed high order elliptic two-point boundary value problems of convection–diffusion type. Under suitable hypotheses, the coercivity of the associated bilinear form is proved and a representation result for the solutions of such problems is given. A family of Galerkin finite element methods based on piecewise polynomial test/trial functions on a Shishkin mesh is constructed and proved to be convergent, uniformly in the perturbation parameter, in energy and W^k_∞ norms. Numerical results are presented for a second order problem and fourth order problems. Full details appear in [2].

FINITE ELEMENT METHODS ON PIECEWISE EQUIDISTANT MESHES FOR INTERIOR TURNING POINT PROBLEMS

Guangfu Sun & Martin Stynes

We consider linear second order singularly perturbed two-point boundary value problems with interior turning points. Piecewise linear Galerkin finite element methods are constructed on various piecewise equidistant meshes designed for such problems. These methods are proved to be convergent, uniformly in the singular perturbation parameter, in a weighted energy norm and the usual L^2 norm. Numerical results are presented. Full details appear in [3].

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Guangfu Sun & Martin Stynes,
Department of Mathematics,
University College,
Cork.

Research Announcements

FINITE ELEMENT ANALYSIS OF EXPONENTIALLY FITTED LUMPED SCHEMES FOR TIME-DEPENDENT CONVECTION-DIFFUSION PROBLEMS

Wen Guo & Martin Stynes

We consider a singularly perturbed linear parabolic initial-boundary value problem in one space variable. Two exponentially fitted schemes are derived for the problem using Petrov-Galerkin finite element methods with various choices of trial and test spaces. On rectangular meshes which are either arbitrary or slightly restricted, we derive global energy norm and L^2 norm and local L^∞ error bounds which are uniform in the diffusion parameter. Numerical results are also presented. Full details appear in [1].

FINITE ELEMENT ANALYSIS OF AN EXPONENTIALLY FITTED NON-LUMPED SCHEME FOR ADVECTION-DIFFUSION EQUATIONS

Wen Guo & Martin Stynes

We analyse a Petrov-Galerkin finite element method for numerically solving an advection-diffusion equation in one space variable. Under reasonable assumptions on the behaviour of the exact solution and a certain stability condition, the scheme is shown to be convergent, uniformly in the diffusion parameter, in global energy and L^2 norms and a local discrete L^∞ norm. Full details appear in [2].

POINTWISE ERROR ESTIMATES FOR A STREAMLINE DIFFUSION SCHEME ON A SHISHKIN MESH FOR A CONVECTION-DIFFUSION PROBLEM

Wen Guo & Martin Stynes

We analyse a streamline diffusion scheme on a special piecewise uniform mesh for a model time-dependent convection-diffusion problem. The method with piecewise linear elements is shown to be convergent, independently of the diffusion parameter, with a pointwise accuracy of almost order 5/4 outside the boundary layer and almost order 3/4 inside the boundary layer. Numerical results are also given. Full details appear in [3].

AN ANALYSIS OF A CELL VERTEX FINITE VOLUME METHOD FOR A PARABOLIC CONVECTION-DIFFUSION PROBLEM

Wen Guo & Martin Stynes

We examine a cell vertex finite volume method which is applied to a model parabolic convection-diffusion problem. By using techniques from finite element analysis, local errors away from all layers are obtained in a seminorm which is related to, but weaker than, the L^2 norm. Full details appear in [4].

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Wen Guo & Martin Stynes,
Department of Mathematics,
University College,
Cork.

Book Review

Around Burnside

Translated from the Russian by James Wiegold

A. I. Kostrikin

Springer-Verlag (Ergeb. Math. Series (3) Vol 20), 220pp.
ISBN 3-540-50602-0
Price $82.00.

Reviewed by Seán Tobin

This is in many ways an extraordinary book, and it is likely to become a collector's item even for those who are not primarily concerned with work on the Burnside problems. Commencing with a curious title, and ending with an eccentric index, it is full of stylistic quirks while also packed with information on the work by Kostrikin and his school which has led recently to the total solution by Efim Zelmanov, [5],[6], of the Restricted Burnside Problem. In his translator's preface Professor Wiegold remarks '...this book has been an interesting challenge to the translator. It is most unusual, in a text of this type, in that the style is racy with many literary allusions and witticisms: not the easiest to translate, but a source of inspiration to continue through material that could daunt by its computational complexity'.

In his preface to this English edition of his book, Professor Kostrikin says, 'Problems of Burnside type have become singularly popular in Moscow and Novosibirsk ... and it is of course advisable for [Russian algebraists] to share their knowledge with Western colleagues'. Our thanks are certainly due to both author and translator for their efforts to make this knowledge available to us, in particular through the medium of this book. Two other recent books worth mentioning in this context are Vaughan-Lee's [4] on the Restricted Problem and Zelmanov's [7] on problems of Burnside type.

Kostrikin's book commences with a brief but careful account of the history of developments in the study of the group problems posed by Burnside in his famous paper of 1902, [1]. These are (in our terminology):

(a) Is a finitely generated periodic group necessarily finite?

(b) Must such a group be finite if the periods are bounded?

(c) If finite, what is the order (in terms of obvious parameters)?

Question (b)–with its corollary (c)–has come to be known as THE Burnside Problem. Having stated (a), Burnside ignored it in his paper, possibly wisely since in 1964 E.S.Golod showed that the answer is "No". Burnside and, later, others obtained positive answers to (b) in a few special cases. If we let $B(n,k)$ represent the quotient of a free group of rank n over the subgroup generated by the k–th powers of all its elements, then by 1958 it was known that for all n, the group $B(n,k)$ is finite when $k = 2$, 3, 4, or 6; and the precise order was known for $k = 2$, 3, or 6 (for $k = 4$, it is still not known). These are the only positive results; but in 1968 Novikov and Adian showed that $B(2,k)$ is infinite for large (> 4380) odd values of k. Since then this bound has been lowered, and other negative results have been obtained, but they do not concern us here.

Already in the nineteen-forties and fifties, a related problem was gaining attention. Even if $B(n,k)$ is infinite for a certain pair n, k, it could well happen that it has a largest finite quotient group $R(n,k)$ such that every finite homomorphic image of $B(n,k)$ is a homomorphic image of $R(n,k)$–equivalently, $B(n,k)$ might have just a finite number of non-isomorphic finite factor-groups. The question of the existence of $R(n,k)$ was called by Magnus, [3], the RESTRICTED Burnside Problem, and it is a problem to which Kostrikin has addressed himself for many years, showing what has been described as 'a heroic capacity for computation'. Incidentally the reviewer attended a course of lectures on Kostrikin's work given by James Wiegold in Canberra during the (Australian) winter of 1965, in which he displayed a similar heroic capacity–which must have stood him in good stead when coping with Kostrikin's book.

Kostrikin published his first result on this problem, a proof that $R(2,5)$ exists, in 1955 and then in 1958 he succeeded in proving that $R(n,p)$ exists for all n and all primes p. The final step, replacing p by any power of p, eluded him despite intense efforts. His book was written to chart the course of this passionate pilgrimage, to point out pitfalls and possible improvements, and to act as a kind of *vade mecum* for others engaged in the same task. And indeed it has done so splendidly, for his co-worker E. Zelmanov, building on the ideas developed by and with Kostrikin, has proved the existence of $R(n,k)$ whenever k is a prime power. Taken in conjunction with the Reduction Theorem proved by P.Hall and G.Higman, [2], in 1956 (modulo certain conjectures about finite simple groups, which have since been confirmed by the Classification Theorem) this has the consequence that $R(n,k)$ exists for all n and all k. Ironically, this result was obtained just as the book under review was being published, and could only be referred to in the translator's preface.

The book is concerned largely with Lie algebras which satisfy certain Engel identities. We will write $[x,y]$ for the product of x and y in a Lie algebra L, and $[x,n.y]$ for the left-normed product $[x,y,...,y]$, where y appears n times. An Engel algebra, or more specifically an $E(n)$–algebra, is a Lie algebra in which the law $[x,n.y] = 0$ is satisfied for some fixed integer n. This is linear in x and y when $n = 1$, but more generally an effort is made to extract multilinear laws (involving terms $[x,u,v,...,w]$) as a consequence, in order to exploit linearity properties of the algebra. The plan of attack on the Restricted problem for prime exponent p is as follows: consider a finitely generated group G with exponent p, and construct an associated Lie ring $L(G)$ using for example the descending central series of G. Then $L(G)$ has characteristic p and may be regarded as an algebra over the integers modulo p. By a theorem of Magnus, this algebra satisfies the Engel identity $[x,(p-1).y] = 0$. Since G is nilpotent, and therefore a finite p-group, if and only if $L(G)$ is nilpotent, the problem now becomes this: show that a finitely generated $E(p-1)$–algebra over the field of p elements is nilpotent, (and calculate its class and order). An explanation of this technique is given in Chapter 7, which surveys

'...the shape of the linear methods in finite group theory that have a bearing on our principal theme...'.

That this is the last chapter rather than the first illustrates the curiously non-linear layout of the book, which skips forward and backwards, looking at the same result reached by different methods, occasionally pointing out cul-de-sacs which had been explored to no avail. Before commenting on the style, however, let us look at the contents.

The first four chapters are devoted to the work on $R(n,p)$ and are intended '...to make available...a text that is easily checked and does not pretend to the deceptive brevity of the original paper'. This is just one of the disarming comments scattered through the text. The author certainly takes pains to get his ideas across, but the checking is not a task for the faint-hearted.

Some shorter alternative proofs are discussed in Chapter 5, while Chapter 6 gives a number of results on nilpotency of other Engel algebras, in particular Razmyslov's theorem on the existence of non-solvable $E(p-2)$–algebras of characteristic p and Zelmanov's theorem on the nilpotency of Engel algebras of characteristic zero. Appendix I gives an argument due to Zelmanov furnishing a recursive bound $f(s,t)$ for the nilpotency class of an s-generator $E(t)$–algebra of characteristic p, where $p > t$. Finally, Appendix II gives a brief biography of Burnside.

The work of Kostrikin and Zelmanov on the Restricted problem depends on studies of 'sandwiches'. An element s in a Lie algebra L is called a 'thin sandwich' if $[x,s,s] = [x,s,y,s] = 0$ for all x and y in L; it is 'thick' if also $[x,s,y,s,z] = 0$ for all x, y and z in L. The lengthy argument for exponent p runs as follows: (i) if L is an $E(n)$–algebra over a field F of characteristic p, where $n < p$, then L contains a (non-zero) thin sandwich; (ii) if L has a thin sandwich, it must also contain a thick sandwich; (iii) if L has a thick sandwich it must also contain a non-trivial abelian ideal; (iv) now let R be the locally nilpotent radical of L, so that L/R is also an $E(n)$–algebra over F. This must be zero, otherwise (iii) is contradicted; thus $L = R$ and we have done.

Now for a few words on the style of this remarkable book. Frankly, this reviewer found the heavy-handed humour and the forced jocular remarks quite tiresome (although, to be fair, the translation of humour is rarely a happy event). On the other hand, the book abounds with interesting and illuminating comments on each chapter; and one must admire an author who can laugh at himself, as for example on page 158: 'Theorem 1.1 refutes a strange conjecture to be found in the survey article [142]'–the reference here is to an article by Kostrikin himself published in 1974. He also pokes a little gentle fun at himself in the Epilogue, where he says: 'The method of sandwiches lies at the heart of this book. Unfortunately, the book itself is written in the form of a sandwich, with layers of similar material in different sections and even in different chapters...Perhaps it would be worthwhile to produce a slight rearrangement...'

Incidentally, the reference here to 'Theorem 1.1' is in fact a misprint, and the text has a small number of obvious misprints. As for the index, it was presumably compiled by a computer and unseen by human eye–how else to account for entries such as 'Bugaboo threat to Lemma 3.1'? There is a fine list of 288 references, and a separate Erratum pamphlet which reproduces this list but with the very useful addition of references to reviews in the Zentralblatt and to English versions (where available) of the many papers in Russian.

To sum up: this book sets out to explain clearly the great contributions of Kostrikin and his Russian collaborators, and to make them available to other workers in the field. It is unusual among books at this level in that the author has stamped his personality upon it, and indeed he comes across as a warm and likeable man. The book succeeds in its purpose, and anyone who wishes to understand the work which has led ultimately to the solution of the Restricted Burnside Problem should acquire a copy of 'Around Burnside'.

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Sean Tobin,
Department of Mathematics,
University College,
Galway.

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