Book Review

Patterns and Waves

The Theory and Applications of Reaction-Diffusion Equations

Peter Grindrod

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Reviewed by Martin Stynes

In Don DeLillo's entertaining novel *White Noise* [1], a central event occurs when the narrator's small U.S. town is threatened by a toxic chemical cloud. Many inhabitants of the town flee. We read [1] "We joined ... the traffic flow into the main route out of town ... the traffic moved in fits and starts".

The traffic moved in fits and starts. We all recognize this phenomenon. As in DeLillo's novel, it occurs even in the absence of traffic lights and stop signs. Apparently all that is needed to trigger the effect is a sufficient density of traffic. Why does it happen? Why does heavy traffic never seem to flow smoothly at constant (albeit low) speed, but instead is subject to speeding up, slowing down and intermittent halting?

We can answer this question after a little analysis. Suppose that the cars are travelling in the direction of the positive x-axis on an infinitely long one-dimensional road. Let $u(x,t)$ denote the car density, which depends both on position x and time t. Here $u = 0$ corresponds to an empty road and $u = 1$ corresponds to maximum congestion.

Let I be any closed and bounded interval on the x-axis. Then the car population of I is $\int_I u\,dx$. The rate at which this population increases in time is given by $\frac{\partial}{\partial t}\left(\int_I u\,dx\right)$. We assume that u is smooth, so this expression equals $\int_I u_t\,dx$.

Assume that our road has neither entrances nor exits. Then changes in the car population of I can result only from cars entering and leaving I along the road. Denoting the car velocity by $v(x,t)$, the net number entering $I = [a,b]$ is given by $-(uv)(b) + (uv)(a)$. Assuming that v is smooth, this equals $-\int_I (uv)_x\,dx$.

Equating that with our earlier formula,

$$\int_I u_t\,dx = -\int_I (uv)_x\,dx.$$

Since I was arbitrary, we conclude that

$$u_t + (uv)_x = 0.$$

It's clear that the velocity v must depend on u. We now make a plausible simplifying assumption, viz., that $v = 1 - u$. Then our differential equation above becomes

$$u_t + (u(1-u))_x = 0,$$

i.e,

$$u_t + (1-2u)u_x = 0.$$

The characteristics of this hyperbolic equation satisfy

$$\frac{dt}{dx} = \frac{1}{1-2u}.$$

The solution u is constant on each characteristic. We see that, as time passes, regions of high car density ($1/2 < u \le 1$) will move in the direction of the negative x-axis. Furthermore, this rate of movement is least for u near $1/2$ and greatest for u near 1. Thus, as regions of higher density move backwards towards regions of lower density, the traffic tends to bunch together and a stop-go regime develops, instead of all cars proceeding at some uniform speed.

The above discussion is a partial answer to Exercise 1.5 on page 63 of the book under review. It illustrates several features of this book: its concern with the understanding and solution of evolutionary nonlinear partial differential equations, their use in modelling phenomena from the real world, and in particular the revelation that smooth initial data can in finite time generate nonsmooth solutions in such models.

In fact, unlike our example, the book (as can be inferred from its title) deals predominately with parabolic partial differential equations. These have the form

$$u_t = \Delta u + f(u, \nabla u, \vec{x}, t).$$

Here t is time, $\vec{x}$ is position in R^n, Δu denotes the Laplacian of u with respect to the variables $\vec{x}$ and f is some nonlinear function.

The author provides a sustained gradual development of concepts and analytical solution techniques. These are introduced fairly painlessly by means of a detailed examination of examples. He succeeds in finding the middle ground between excessive detail and inadequate explanations. Nevertheless, the reader is clearly expected to use pen and paper to verify various claims. I did not check many calculations in detail, but, for example, the analysis on p. 216 contains several typographical errors. This seems to have been a momentary atypical lapse.

The presentation is nicely structured. Technical difficulties are hived off to clearly marked subsections, so the overall flow of the book is not impeded. Internal cross-referencing, both forwards and backwards, is of an exceptionally high standard. Chapter 1, which occupies about one-quarter of the book, introduces the basic ideas and techniques. The other four Chapters deal (in order of increasing complexity) with observable phenomena in the solutions of nonlinear evolutionary partial differential equations. Their titles are: 2 – Pattern formation, 3 – Plane waves, 4 – A geometrical theory for waves, 5 – Nonlinear dispersal mechanisms. Applications, mostly from physiology, biology and chemistry, are scattered throughout these pages. For example, the Belousov-Zhabotinsky cyclic chemical reaction with its spectacular spiral wave patterns is the main example discussed in Chapter 4.

The author resists the temptation to give excessive attention to side issues. Instead, adequate references are given for further study of specific topics. A significant omission is the almost

complete absence of discussion and references for the numerical solution of the problems examined in the book. This is surprising since several Figures in the book have been produced by numerical computation of approximate solutions; if the author believes (by implication) that such computed solutions are instructive, then he should make some attempt to provide guidelines to the reader who wishes to perform numerical experiments. As the differential equations under consideration are nonlinear and consequently can be difficult to analyse, a computational approach may in practice often be an attractive option.

The book is suitable for beginning postgraduate students in applied mathematics, but pays more than lip service to basic theoretical issues such as local existence criteria. It is more accessible than Smoller's "purer" text [2]. I would welcome the opportunity to teach a course with Grindrod's book as the main text, supplemented by a little material from numerical analysis.

References

[1] D. DeLillo, White Noise (Picador Edition). Pan Books: London, 1986.

[2] J. A. Smoller, Shock Waves and Reaction Diffusion Equations. Springer-Verlag: New York, 1983.

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