[4] M. A. Kaashoek, Closed Linear Operators on Banach Spaces: Amsterdam, 1963.

[5] J. Lindenstrauss and L. Tzafriri, *On the Complemented Subspaces Problem*, Israel J. Math. 9 (1971), 263–9.

Mícheál Ó Searcóid,
Roinn na Matamaitice,
Coláiste na hOllscoile,
Baile Átha Cliath.

BOUNDARY BEHAVIOUR OF HOLOMORPHIC AND HARMONIC FUNCTIONS*

Stephen J. Gardiner

Abstract: We give below a survey of some recent results concerning the boundary behaviour of holomorphic and harmonic functions. The unifying theme is the role played by the integral condition

$$\int_{-1}^{1} \frac{\phi(t)}{t^2} dt < \infty, \qquad (1)$$

where ϕ is a non-negative Lipschitz function.

1. Thin sets

Let Ω be a domain (non-empty, connected open set) in the complex plane $\mathbb{C}$. Recall that a function $u : \Omega \to (-\infty, \infty]$, where $u \not\equiv \infty$, is called *superharmonic* on Ω if u is lower semicontinuous, i.e.,

$$u(z_0) \le \liminf_{z \to z_0} u(z) \qquad (z_0 \in \Omega), \qquad (2)$$

and if

$$u(z_0) \ge \int_0^{2\pi} u(z_0 + re^{i\theta}) \frac{d\theta}{2\pi} \text{ when } \{z : |z - z_0| \le r\} \subset \Omega. \qquad (3)$$

*This article is based on a lecture delivered at the 44th British Mathematical Colloquium held at the University of Strathclyde in April 1992.

As the definition suggests, such functions need not be continuous. However (assuming that $0 \in \Omega$ for simplicity) we can combine (2), (3) and Fatou's lemma to obtain

$$u(0) \leq \int_0^{2\pi} \liminf_{r\to 0+} u(re^{i\theta})\frac{d\theta}{2\pi} \leq \liminf_{r\to 0+} \int_0^{2\pi} u(re^{i\theta})\frac{d\theta}{2\pi}$$
$$\leq \limsup_{r\to 0+} \int_0^{2\pi} u(re^{i\theta})\frac{d\theta}{2\pi} \leq u(0),$$

so

$$\int_0^{2\pi} u(re^{i\theta})\frac{d\theta}{2\pi} \to u(0) \qquad (r \to 0+).$$

Thus superharmonic functions possess a certain weak, or "average", continuity property. More specifically, it can be asserted that

$$u(z) \to u(0) \qquad (z \to 0, z \notin E),$$

where the exceptional set E is "thin" at 0. As an upper bound on how much of E exists near 0, we mention that $E\backslash\{0\}$ is contained in an open set whose circular projection, F, onto the interval (0,1) satisfies

$$\sum_{k=1}^{\infty} \frac{k}{-\log|\{t \in F : 2^{-k-1} < t < 2^{-k}\}|} < \infty.$$

(Here $|A|$ denotes the one-dimensional Lebesgue measure of A.) On the other hand, E can be highly dispersed: for example, E can be dense in Ω.

Formally, a set E is called *thin at* 0 if one of the following (equivalent) conditions holds:

(i) there is a superharmonic function u on a neighbourhood of 0 such that

$$\liminf_{z\to 0, z\in E} u(z) > \liminf_{z\to 0} u(z) \; ; \tag{4}$$

(ii) there is a superharmonic function u on a neighbourhood of 0 such that

$$\liminf_{z\to 0, z\in E} \frac{u(z)}{\log 1/|z|} > \liminf_{z\to 0} \frac{u(z)}{\log 1/|z|} \; ;$$

(iii) $\sum_{k=1}^{\infty} kC^*(\{z \in E : 2^{-k-1} \leq |z| \leq 2^{-k}\}) < \infty$

(Wiener's criterion),

where $C^*(A)$ denotes the outer capacity (see [20, Chapter 7]) of a set A with respect to the unit disc.

With regard to condition (i) above, we remark that the right hand side of (4) is equal to $u(0)$. Thus sets E which are thin at 0 are characterized by the property that knowledge of the values of u on $E\backslash\{0\}$ is not sufficient to determine $u(0)$. An account of thin sets can be found in Helms [20, Chapter 10].

There is a corresponding notion of thinness at a boundary point that can be defined by analogy to (ii) above. Let $D_0 = \{x + iy : y > 0\}$ and define

$$P(z) = \frac{1}{\pi}\frac{y}{x^2 + y^2} \qquad (z = x + iy \in D_0).$$

(This is the Poisson kernel for D_0 with pole at 0). A subset E of D_0 is called *minimally thin at* 0 *with respect to* D_0 if there is a positive superharmonic function u on D_0 such that

$$\liminf_{z\to 0, z\in E} \frac{u(z)}{P(z)} > \liminf_{z\to 0, z\in D_0} \frac{u(z)}{P(z)}. \tag{5}$$

Again minimally thin sets may be dense (in D_0), and can only be described in terms of capacities. However, if we are dealing solely with harmonic functions u, the sets E that can arise in (5) are of a more specific nature due to Harnack's inequalities. A precise description of such sets is given below in a reformulation of a result of Beurling [3].

Theorem A. *The following are equivalent conditions on a subset E of D_0:*

(i) *there is a positive harmonic function h on D_0 such that*

$$\liminf_{z\to 0, z\in E} \frac{h(z)}{P(z)} > \liminf_{z\to 0, z\in D_0} \frac{h(z)}{P(z)} \ ;$$

(ii) *there is a positive number ϵ and a Lipschitz function $\phi : [-1,1] \to [0,\infty)$ such that*

$$\int_{-1}^{1} \frac{\phi(t)}{t^2} dt < \infty \quad \text{and } E \cap \{|z|<\epsilon\} \subseteq \{x+iy : 0<y<\phi(x)\}.$$

In the following sections we discuss several applications of the above integral condition.

2. The angular derivative problem

In this section D denotes a simply connected domain in $\mathbb{C}$ such that $0 \in \partial D$. Further, f denotes a bijective holomorphic mapping from D_0 to D which has angular limit 0 at 0. (We recall that a function g on D_0 is said to have *angular limit* l at 0 if, for any positive number k,

$$g(z) \to l \qquad (z = x+iy \to 0 \ , \ y > k|x|).)$$

If the derivative f' has an angular limit at 0, this is called the *angular derivative of f at* 0, and is denoted by $f'(0)$. The existence of $f'(0)$ depends on D, but not on the choice of f: that is, if it exists for one such function f then it exists for them all. For further properties of the angular derivative we refer to Pommerenke [21, Chapter 10]. The angular derivative problem is as follows: *give necessary and sufficient geometric conditions on D such that $f'(0)$ exists, and $0 < |f'(0)| < \infty$.*

This problem has a long history and remains unsolved. However, significant progress was made recently by Burdzy [7], using deep probablistic methods. To state his result, we define F_ϵ to be the family of functions $\phi : \mathbb{R} \to \mathbb{R}$ which satisfy the Lipschitz condition $|\phi(x) - \phi(y)| \le |x-y|$ and for which $\{y > \phi(x)\} \cap \{|z|<\epsilon\} \subseteq D \cap \{|z|<\epsilon\}$. Further, let

$$\phi_\epsilon(t) = \inf_{\phi\in F_\epsilon} \phi(t), \quad \phi_\epsilon^+ = \max\{\phi_\epsilon, 0\}, \quad \phi_\epsilon^- = \max\{-\phi_\epsilon, 0\}.$$

Burdzy's theorem is stated below.

Theorem 1. *Suppose that, for some $\epsilon > 0$,*

$$\int_{-1}^{1} \frac{\phi_\epsilon^+(t)}{t^2} dt < \infty.$$

Then $f'(0)$ exists and $0 \le |f'(0)| < \infty$. Further, $f'(0) \ne 0$ if and only if

$$\int_{-1}^{1} \frac{\phi_\epsilon^-(t)}{t^2} dt < \infty.$$

Rodin and Warschawski [22] attempted to prove Theorem 1 by classical means, but were only partly successful: the problem was to find a classical proof of Theorem 2 below, originally proved by Burdzy and Williams [8] using probabilistic methods. This was first achieved by Carroll [11] using an ingenious, but very difficult, argument. Since then two short proofs of the result have been found: one by Sastry [23] based on extremal length arguments, and one by the author [15], based on Beurling's Theorem A. Let $\phi : \mathbb{R} \to \mathbb{R}$ be Lipschitz, and let $D_\phi = \{x+iy : y > \phi(x)\}$.

Theorem 2. *Let $\epsilon > 0$ and let h_ϕ be a positive harmonic function on $D_\phi \cap \{|z|<\epsilon\}$ which continuously vanishes on $\partial D_\phi \cap \{|z|<\epsilon\}$. If*

$$\int_{-1}^{1} \frac{\phi^+(t)}{t^2} dt < \infty \quad \text{and} \quad \int_{-1}^{1} \frac{\phi^-(t)}{t^2} dt = \infty, \tag{6}$$

then $h_\phi(iy)/y \to \infty$ as $y \to 0+$.

The idea of the proof in [15] is to use Theorem A to compare positive harmonic functions $h_0, h_{\phi+}, h_\phi$ on the regions $D_0, D_{\phi+}, D_\phi$ (resp.) which vanish on the boundary, at least near 0. It is easy to see that $h_0(iy)/y$ has a positive limit as $y \to 0+$. The same can be established for $h_{\phi^+}(iy)/y$ using the convergent integral condition in (6). However, the "negative humps" in the boundary of D_ϕ cause $h_\phi(iy)/y$ to diverge to ∞ as $y \to 0+$ because of the divergent integral in (6). This is the tricky part of the proof, as Theorem A does not immediately apply to these "negative humps". See [15] for further details.

3. X-domains

Let U be the unit disc, X be a certain class of holomorphic functions on U, and $\mathcal{H}(U,D)$ be the class of all holomorphic functions $f : U \to D$, where D is some domain in $\mathbb{C}$. In this section we are interested in results of the form: *$f \in X$ for all f in $\mathcal{H}(U,D)$ if and only if D satisfies certain geometric conditions.* For example, X could be the Nevanlinna class $\mathcal{N}$ of holomorphic functions f on U for which

$$\sup_{0<r<1} \int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta < \infty \ ;$$

or the Smirnov class $\mathcal{N}^+$ of functions f in $\mathcal{N}$ for which

$$\int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta \to \int_0^{2\pi} \log^+ |f(e^{i\theta})| d\theta \qquad (r \to 1-).$$

(Any function f in $\mathcal{N}$ has radial boundary values $f(e^{i\theta})$ almost everywhere.) The following two results are due to Frostman [14] and Ahern and Cohn [1] respectively. A set is called *thin at ∞* if its inversion in the unit circle is thin at 0.

Theorem B. *Let D be a domain in $\mathbb{C}$. Then $f \in \mathcal{N}$ for all f in $\mathcal{H}(U,D)$ if and only if ∂D has positive logarithmic capacity.*

Theorem C. *Let D be a domain in $\mathbb{C}$. Then $f \in \mathcal{N}^+$ for all f in $\mathcal{H}(U,D)$ if and only if $\mathbb{C}\backslash D$ is not thin at ∞.*

We will present two further results of this type. Let h^1 denote the class of harmonic functions on U which can be written as the difference of two positive harmonic functions on U.

Theorem 3. *Let D be a simply connected domain which contains $\{x+iy : x > 0\}$. Then $\Re f \in h^1$ for all f in $\mathcal{H}(U,D)$ if and only if*

$$\int_{-\infty}^{\infty} \frac{\text{dist}\,(iy, \partial D)}{1+y^2} dy < \infty. \qquad (7).$$

We observe that, if $D \subseteq \{x > 0\}$ and $f \in \mathcal{H}(U,D)$, then $\Re f > 0$, so (trivially) $\Re f \in h^1$. Theorem 3 shows precisely how much larger than $\{x > 0\}$ we can allow D to be while still ensuring that $\Re f \in h^1$. The condition (7) is of the same type as (1), after an inversion in the unit circle.

It is easily seen that $\Re f \in h^1$ if and only if $e^f \in \mathcal{N}$. Referring back to Theorems B and C we are led to consider when $e^f \in \mathcal{N}^+$. A subset of $\{x > 0\}$ is called minimally thin at ∞ if its inversion in the unit circle is minimally thin at 0.

Theorem 4. *Let D be as in Theorem 3, suppose (7) holds, and let D_1 be a domain contained in D. Then $e^f \in \mathcal{N}^+$ for all f in $\mathcal{H}(U,D_1)$ if and only if $\{x > 0\}\backslash D_1$ is not minimally thin at ∞ with respect to $\{x > 0\}$.*

Here D_1 is not required to be simply connected. The larger is the set $D\backslash D_1$, the smaller is $\mathcal{H}(U,D_1)$. Theorem 4 describes precisely how large $D\backslash D_1$ must be to ensure that we have the stronger property $e^f \in \mathcal{N}^+$ for all f in $\mathcal{H}(U,D_1)$. It turns out that only $\{x > 0\}\backslash D_1$ is significant. We remark in passing that $\{x > 0\}\backslash D_1$ is not minimally thin at ∞ with respect to $\{x > 0\}$ if and only if the set

$$\left\{(x_1, \cdots, x_4) \in \mathbb{R}^4 : (x_1^2 + x_2^2 + x_3^2)^{1/2} + ix_4 \notin D_1\right\}$$

is not thin at infinity in $\mathbb{R}^4$. Theorems 3 and 4 are proved in [16]. Theorem 3 is related to the angular derivative problem.

4. Sets of determination for harmonic functions

Let $P(\zeta, z)$ denote the Poisson kernel for U with pole ζ; that is,

$$P(\zeta, z) = \frac{1}{2\pi} \cdot \frac{1-|z|^2}{|z-\zeta|^2} \qquad (z \in U, \zeta \in \partial U).$$

If ζ is a fixed point of ∂U, then $P(\zeta, \cdot)$ is a positive harmonic function on U which vanishes on $\partial U \setminus \{\zeta\}$. However, in what follows, we will sometimes fix z and regard $P(\cdot, z)$ as a positive continuous function on ∂U. Let $\mathcal{H}^+$ denote the collection of positive harmonic functions on U. There is a one-to-one correspondence between members h of $\mathcal{H}^+$ and finite Borel measures μ_h on ∂U, given by

$$h(z) = \int_{\partial U} P(\zeta, z) d\mu_h(\zeta) \qquad (z \in U).$$

We consider here two seemingly different types of problem:

(i) given a class A of harmonic functions on U, characterize those subsets E of U such that $\sup_E H = \sup_U H$ for all H in A;
(ii) given a class B of functions on ∂U, characterize those subsets E of U such that any f in B has the form $f = \sum_1^\infty \lambda_k P(\cdot, z_k)$, where the points z_k belong to E.

Surprisingly there is a close relationship between these two types of problem. The key idea in both is that there must be "enough of E" near "appropriate points" of ∂U. We define a set

$$E_{1/2} = \bigcup_{w \in E} \{z : |z - w| < (1 - |w|)/2\}$$

and a function

$$E_{1/2}^*(\zeta) = \int_{E_{1/2}} |z - \zeta|^{-2} dx dy \qquad (\zeta \in \partial U),$$

which takes values in $[0, \infty]$. By "enough of E" near ζ we mean $E_{1/2}^*(\zeta) = \infty$.

Theorem 5. *Let $E \subseteq U$. The following are equivalent:*
(i) $\sup_E H = \sup_U H$ *for every H in h^1;*
(ii) $E_{1/2}^*(\zeta) = \infty$ *for every ζ in ∂U;*
(iii) *for every positive continuous function f on ∂U there exist a sequence (λ_k) of positive numbers and a sequence (z_k) of points in E such that*

$$f(\zeta) = \sum_{k=1}^{\infty} \lambda_k P(\zeta, z_k) \qquad (\zeta \in \partial U). \tag{8}$$

This elegant result is due to Hayman and Lyons [19]. The convergence in (8) is uniform, by Dini's theorem. Alternative proofs and a variety of extensions can be found in [5], [13], [17], [12] and [2]. In particular, [17] contains a short proof based on Beurling's Theorem A (cf. the definition of $E_{1/2}^*(\zeta)$) together with a result which includes the following.

Theorem 6. *Let $E \subseteq U$ and $h \in \mathcal{H}^+$. The following are equivalent:*
(i) $\inf_E H/h = \inf_U H/h$ *for all H in $\mathcal{H}^+$;*
(ii) $E_{1/2}^*(\zeta) = \infty$ *for almost every (μ_h) ζ in ∂U.*
Further, each of the above conditions implies:
(iii) *for every f in $L^1(\mu_h)$ there exist (λ_k) in $\ell_1(\mathbb{C})$ and a sequence (z_k) of points in E such that*

$$f = \sum_{k=1}^{\infty} \lambda_k P(\cdot, z_k)/h(z_k) \tag{9}$$

(convergence in the sense of $L^1(\mu_h)$), and

$$\|f\|_{L^1(\mu_h)} = \inf\{\Sigma|\lambda_k| : (9) \text{ holds for some } (z_k) \text{ in } E\}.$$

If $h \equiv 1$, then (cf. Bonsall [4]) (iii) above is actually *equivalent* to (i), (ii) and:

(ii′) *for almost every (Lebesgue) ζ in ∂U, there is a sequence of points in E which converges to ζ within some angle.*

5. Better-than-angular limits

Sections 2 and 3 illustrated the relevance of (1) to boundary distortion. In section 4, divergence of the integral in (1) was related to having "enough" of E to determine suprema/infima of harmonic functions on U, or to achieve representation of functions on ∂U in terms of Poisson kernels. Finally, in this section, we discuss the role of (1) in describing approach regions for boundary behaviour of holomorphic and harmonic functions.

Many results in function theory state that functions (in D_0, say) have angular limits almost everywhere on ∂D_0, or on a subset of ∂D_0 of positive Lebesgue measure. We will now point out that rather more can be asserted. Let Φ denote the class of Lipschitz functions $\phi : \mathbb{R} \to [0, \infty)$ such that (1) holds. A function g on D_0 is said to have *Φ-limit l at $t \in \mathbb{R}$* if there exists ϕ in Φ such that

$$g(z) \to l \qquad (z \to t,\ y > \phi(x - t)).$$

It is easy to see that the existence of a Φ-limit at t implies that g has an angular limit at t, but not conversely.

Theorem 7. *Let u be a harmonic function on D_0 such that, for every t in E (where $E \subseteq \mathbb{R}$), there is an angle with vertex at t in which u is bounded below. Then u has (finite) Φ-limits at almost every (Lebesgue) t in E.*

The existence of angular limits under the above hypothesis is due to Calderón [9] and Carleson [10]. Brelot and Doob [6] showed that u must have minimal fine limits at almost every point t in E. However, the latter result does not immediately combine with Theorem A to yield Theorem 7, since u need not be positive on D_0. The proof of Theorem 7 can be found in [15]. An example of its application occurs in [18]. The conclusion of the theorem clearly remains true if u is a holomorphic function on D_0 which, for each t in E, is bounded in an angle with vertex at t.

NOTE. The results in this paper which concern harmonic functions have natural analogues in $\mathbb{R}^n (n \geq 3)$. Details can be found in the appropriate references.

References

[1] P. Ahern and W. Cohn, *A geometric characterization of N^+ domains,* Proc. Amer. Math. Soc. **88** (1983), 454–8.

[2] H. Aikawa, *Sets of determination for harmonic functions in NTA domains* (preprint).

[3] A. Beurling, *A minimum principle for positive harmonic functions,* Ann. Acad. Sci. Fenn. Ser. AI **372** (1965), 1–7.

[4] F. F. Bonsall, *Domination of the supremum of a bounded harmonic function by its supremum over a countable subset,* Proc. Edinburgh Math. Soc. **30** (1987), 471–7.

[5] F. F. Bonsall and D. Walsh, *Vanishing ℓ^1-sums of the Poisson kernel, and sums with positive coefficients,* Proc. Edinburgh Math. Soc. **32** (1989), 431–47.

[6] M. Brelot and J. L. Doob, *Limites angulaires et limites fines,* Ann. Inst. Fourier Grenoble **13** (1963), 395–415.

[7] K. Burdzy, *Brownian excursions and minimal thinness. Part III: Applications to the angular derivative problem,* Math. Z. **192** (1986), 89–107.

[8] K. Burdzy and R. J. Williams, *On Brownian excursions in Lipschitz domains. Part I. Local path properties,* Trans. Amer. Math. Soc. **298** (1986), 289–305.

[9] A. P. Calderón, *On the behaviour of harmonic functions at the boundary,* Trans. Amer. Math. Soc. **68** (1950), 47–54.

[10] L. Carleson, *On the existence of boundary values for harmonic functions in several variables,* Ark. Mat. **4** (1962), 393–9.

[11] T. F. Carroll, *A classical proof of Burdzy's theorem on the angular derivative,* J. London Math. Soc. (2) **38** (1988), 423–41.

[12] N. Dudley-Ward, Doctoral thesis, University of York, 1991.

[13] M. Essén, *On minimal thinness, reduced functions and Green potentials* (preprint).

[14] O. Frostman, *Potentiels d'équilibre et capacité des ensembles avec quelques applications à la théorie des fonctions,* Med. Lunds Univ. Mat. Sem. **3** (1935), 1–118.

[15] S. J. Gardiner, *A short proof of Burdzy's theorem on the angular derivative,* Bull. London Math. Soc. **23** (1991), 575–9.

[16] S. J. Gardiner, *Positive harmonic majorization of the real part of a holomorphic function,* Proc. Amer. Math. Soc. (to appear).

[17] S. J. Gardiner, *Sets of determination for harmonic functions,* Trans. Amer. Math. Soc. (to appear).

[18] S. J. Gardiner, *Angular limits of holomorphic functions which satisfy an integrability condition,* Mh. Math. (to appear).

[19] W. K. Hayman and T. J. Lyons, *Bases for positive continuous functions,* J. London Math. Soc. (2) **42** (1990), 292–308.

[20] L. L. Helms, Introduction to potential theory. Krieger: New York, 1975.

[21] C. Pommerenke, Univalent functions. Vandenhoeck and Ruprecht: Göttingen, 1975.

[22] B. Rodin and S. Warschawski, *Remarks on a paper of K. Burdzy,* J. Analyse Math. **46** (1980), 251–60.

[23] S. Sastry, *An extremal length proof of Burdzy's theorem* (preprint).

Stephen J. Gardiner
Department of Mathematics
and Statistics
McGill University
Montréal
Québec
Canada H3A 2K6

Permanent address:
Department of Mathematics
University College
Dublin 4
Ireland

INTERNAL FORCING AXIOMS: MARTIN'S AXIOM AND THE PROPER FORCING AXIOM

Dedicated to the memory of Alan H. Mekler.

Eoin Coleman

In the course of the last twenty-five years research in the combinatorics of partially ordered sets has resulted in the discovery of new set-theoretic hypotheses — sometimes dubbed internal forcing axioms. This elementary article presents in section 1 the simplest of these (Martin's Axiom). In section 2 we look at some applications (the completeness of the category ideal, Lusin sets, Q-sets, problems of Moore, Alexandroff, Suslin, Whitehead and Kaplansky). Finally in section 3 we deal briefly with the Proper Forcing Axiom, a powerful generalization of Martin's Axiom. We've collected the relevant references in an annotated bibliography in section 4, rather than in the body of the text.

We try to show concretely how internal forcing axioms work (giving complete proofs whenever feasible), stressing the resemblance to the classical diagonal arguments of Baire and Cantor. In our choice of applications we seek to underline the fact that mathematical conjectures having no apparent set-theoretic reference may depend for their resolution on axioms beyond those of ordinary set theory. To put it another way, there are at least three truth values in mathematics: true, false, and independent of ordinary set theory.

Section 1: Forcing

Internal forcing axioms are about forcings. Let us recall that a *forcing* is simply a partial order, i.e. a pair $\mathbf{P} = (P, \leq)$ such that P is a non-empty set, $\leq$ is a reflexive antisymmetric transitive