closed with an address by the President of the IMS, Dr Richard Timoney.

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THE CONTINUITY OF THE SEMI-FREDHOLM INDEX

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Introduction

It is well-known and easy to prove that the index function is continuous on the set of Fredholm operators on a Banach space. It is also true that the index function is continuous on the larger set of semi-Fredholm operators. The proof presents no difficulty in the case where the semi-Fredholm operators are simply those operators which are left or right invertible modulo the compact operators, as happens in the case where the Banach space is a Hilbert space. The usual proofs in the more general context [4, p.60], [1, pp.62-63] use the notion of gap between between subspaces and require more preliminary work than one might have suspected necessary. In this note we show how to avoid such unsatisfactory excursions by giving a natural operator-theoretic proof of this basic result. The nature of the proof makes it convenient to consider the slightly more general case in which the operators act between two possibly different spaces.

1. Preliminaries

If X and Y are Banach spaces, then $\mathcal{B}(X,Y)$ will denote the set of bounded linear operators from X to Y, and $\mathcal{F}(X,Y)$ will denote the set of finite rank operators in $\mathcal{B}(X,Y)$. When $X = Y$ we shall write $\mathcal{B}(X)$ instead of $\mathcal{B}(X,Y)$ and $\mathcal{F}(X)$ instead of $\mathcal{F}(X,Y)$; all similar notation will be abbreviated in the same way. Identity operators on spaces will be denoted by I; the space in question will always be obvious from the context. For each $T \in \mathcal{B}(X,Y)$, the nullity $\mathrm{nul}(T)$, and the defect $\mathrm{def}(T)$ of T are defined to be the dimension of the nullspace of T and the codimension in Y of the closure of the range of T respectively. Provided not both these quantities are infinite, we define the index of T by

$$\mathrm{ind}(T) = \mathrm{nul}(T) - \mathrm{def}(T).$$

We shall be concerned with two sets of closed range operators, the sets of upper and lower semi-Fredholm operators, defined respectively to be

$\Phi_+(X,Y) = \{T \in \mathcal{B}(X,Y) : T(X) \text{ closed in } Y \text{ and } \mathrm{nul}(T) < \infty\}$
and
$\Phi_-(X,Y) = \{T \in \mathcal{B}(X,Y) : T(X) \text{ closed in } Y \text{ and } \mathrm{def}(T) < \infty\}$.

Their intersection, denoted by $\Phi(X,Y)$, is the set of Fredholm operators, and their union, $\Phi_\pm(X,Y)$, is the set of semi-Fredholm operators. We shall consider two further subsets of $\mathcal{B}(X,Y)$, called the sets of left Fredholm and right Fredholm operators. These are denoted by $\Phi_l(X,Y)$ and $\Phi_r(X,Y)$ and are defined by

$\Phi_l(X,Y) = \{T \in \mathcal{B}(X,Y) : \exists S \in \mathcal{B}(Y,X) \text{ with } ST - I \in \mathcal{F}(X)\}$
and
$\Phi_r(X,Y) = \{T \in \mathcal{B}(X,Y) : \exists S \in \mathcal{B}(Y,X) \text{ with } TS - I \in \mathcal{F}(Y)\}$.

It is well known that $\Phi_l(X,Y)$ is contained in $\Phi_+(X,Y)$ and $\Phi_r(X,Y)$ is contained in $\Phi_-(X,Y)$ and that, for arbitrary Banach spaces, these inclusions are often proper. Indeed, unless Y is a Hilbert space, there always exists a Banach space X such that the inclusions are proper [5]. Our strategy is to reduce the general continuity result to that for left and right Fredholm operators. The latter result is easy and we prove it in this section for the sake of completeness.

Note first that the following index theorem holds just as it does for Fredholm operators on a single space, with the same proof [2 pp.208–9]: For Banach spaces X, Y, Z and operators $T_1 \in \Phi_l(X,Y)$ and $T_2 \in \Phi_l(Y,Z)$ we have $T_2T_1 \in \Phi_l(X,Z)$ and $\mathrm{ind}(T_2T_1) = \mathrm{ind}(T_2) + \mathrm{ind}(T_1)$, with a similar result for right Fredholm operators.

Lemma 1.1. *Let X and Y be Banach spaces. Let $T \in \Phi_l(X,Y)$ and $G \in \mathcal{F}(X,Y)$. Then $T+G \in \Phi_l(X,Y)$ and $\mathrm{ind}(T+G) = \mathrm{ind}(T)$.* (A similar result holds for right Fredholm operators).

Proof: It is obvious that $T+G \in \Phi_l(X,Y)$. Let $P \in \mathcal{B}(Y)$ be any projection of Y onto $G(X)$. Then $(I-P)(T+G) = (I-P)T$ and, since the index of $I-P$ is zero, the result follows from the index theorem. □

Now, since the group of invertible elements of $\mathcal{B}(Y)$ is open, the continuity result for left and right Fredholm operators is a special case of the following proposition:

Proposition 1.2. *Let X and Y be Banach spaces and suppose $T \in \Phi_l(X,Y)$. Suppose $S \in \mathcal{B}(Y,X)$ and $G \in \mathcal{F}(X)$ satisfy $ST - I = G$. Suppose $U \in \mathcal{B}(X,Y)$ is such that $I+US$ is Fredholm of index zero in $\mathcal{B}(Y)$. Then $T+U \in \Phi_l(X,Y)$ and the index of $T+U$ is the same as the index of T.* (The corresponding result with the obvious changes holds for right Fredholm operators).

Proof: Since $T + U = (I+US)T - UG$, the result follows from the index theorem and Lemma 1.1. □

There is a lemma attributed by Banach to Auerbach which states that if X is a finite dimensional normed linear space of dimension n and X^* is its dual, then there exist normalized bases $x_1,\ldots,x_n$ and $f_1,\ldots,f_n$ for X and X^* respectively such that $f_i(x_j) = 1$ if $i=j$ and $f_i(x_j) = 0$ otherwise $(0 \le i,j \le n)$. Ruston's delightfully simple deduction of this result from the compactness of the unit ball can be found in [3, p.200]. The following easy corollaries are given on pages 312–314 of the same work.

Lemma 1.3. *Suppose X is a normed linear space and Y and Z are closed subspaces of X with $\dim(Y) = n$ and $\dim(X/Z) = m$ (m and n finite). Let $\epsilon > 0$. Then there exist projections $P, Q \in \mathcal{B}(X)$ with $P(X) = Y$ and $(I-Q)(X) = Z$ such that $\|P\| \le n$ and $\|Q\| \le m + \epsilon$.*

2. Results

Lemma 2.1. *Let X and Y be Banach spaces. Then the sets* $\{T \in \mathcal{B}(X,Y) : T \text{ is bounded below and } \operatorname{ind}(T) = -\infty\}$ *and* $\{T \in \mathcal{B}(X,Y) : T \text{ is surjective and } \operatorname{ind}(T) = \infty\}$ *are open in* $\mathcal{B}(X,Y)$.

Proof: We consider only the first set; the other can be treated similarly. Recall that the set of bounded below operators in $\mathcal{B}(X,Y)$ is open and that such operators are characterized as the closed range operators with zero nullity. Let $T \in \mathcal{B}(X,Y)$ be a bounded below operator and let $\epsilon > 0$ be such that for each $U \in \mathcal{B}(X,Y)$ with $\|U\| < \epsilon$ we have $T + U$ bounded below. Let $(T_n)_{n\in\mathbf{N}}$ be a sequence in $\mathcal{B}(X,Y)$, converging to T, such that $\|T - T_n\| < \epsilon$ for all $n \in \mathbf{N}$ and suppose $\operatorname{ind}(T_n) > -\infty$ for all $n \in \mathbf{N}$. Then $T_n \in \Phi(X,Y)$ for each $n \in \mathbf{N}$. We must show that $\operatorname{ind}(T) \neq -\infty$. Suppose firstly that the T_n all have the same finite index, $-k$. We shall justify this assumption later. Then $\operatorname{def}(T_n) = k < \infty$ $(n \in \mathbf{N})$, and, by Lemma 1.3, there exists a sequence of projections $(Q_n)_{n\in\mathbf{N}}$ in $\mathcal{F}(Y)$ such that both $T_n(X) = (I - Q_n)(Y)$ and $\|Q_n\| \leq k + \epsilon$ for all $n \in \mathbf{N}$.

Now, for each $n \in \mathbf{N}$, there exists $S_n \in \mathcal{B}(Y,X)$ such that

$$S_nT_n = I \quad \text{and} \quad T_nS_n = I - Q_n.$$

It follows from this that, for each $n \in \mathbf{N}$,

$$\frac{TS_n}{\|S_n\|} = \frac{(T - T_n)S_n}{\|S_n\|} + \frac{I - Q_n}{\|S_n\|}.$$

Since the $I - Q_n$ are uniformly bounded and since T is bounded below, it follows that the S_n are uniformly bounded. So $\|S_n(T - T_n)\| < 1$ for sufficiently large $n \in \mathbf{N}$, and, since $S_nT = I + S_n(T - T_n)$, it follows that S_nT is invertible in $\mathcal{B}(X)$. Hence there exists $S \in \mathcal{B}(Y,X)$ such that $ST = I$. In particular, $T \in \Phi_l(X,Y)$ so that $\operatorname{ind}(T) = -k \neq -\infty$, by Proposition 1.2.

Now our hypothesis that the T_n all have the same index is actually true: Suppose $m, n \in \mathbf{N}$ and let

$$\beta = \sup\{\alpha \in [0,1] : \operatorname{ind}(\alpha T_n + (1-\alpha)T_m) = \operatorname{ind}(T_m)\}.$$

Write $V = \beta T_n + (1-\beta)T_m$. Then V is bounded below, since $\|T + V\| < \epsilon$, and is the limit of a sequence of Fredholm operators each of whose index is $\operatorname{ind}(T_m)$. By what we have just proved, putting V in place of T, we get $\operatorname{ind}(V) = \operatorname{ind}(T_m)$. Now the openness of $\Phi(X,Y)$ and the continuity of the Fredholm index proved in 1.2 ensure that $\beta = 1$ and that $\operatorname{ind}(T_n) = \operatorname{ind}(T_m)$. This completes the proof. □

Theorem 2.2. *$\Phi_\pm(X,Y)$ is an open set and the index is continuous on $\Phi_\pm(X,Y)$.*

Proof: We prove the result for $\Phi_+(X,Y)$. That for $\Phi_-(X,Y)$ can be proved similarly. Suppose then that $T \in \Phi_+(X,Y)$. Since the result for $\Phi(X,Y)$ is contained in Proposition 1.2, we may assume that $\operatorname{ind}(T) = -\infty$. The nullspace Z of T is finite dimensional, so has a complement W in X. Denote by $T_W : W \to Y$ the restriction of T to W. Then T_W is bounded below and also $\operatorname{ind}(T_W) = -\infty$. By Lemma 2.1, there exists $\epsilon > 0$ such that for each $U \in \mathcal{B}(X,Y)$ with $\|U\| < \epsilon$ we have $(T+U)_W \in \mathcal{B}(W,Y)$ bounded below and $\operatorname{ind}(T+U)_W = -\infty$. Since Z is finite dimensional, it follows that $T + U \in \Phi_+(X,Y)$ and $\operatorname{ind}(T+U) = -\infty$ as required. □

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Roinn na Matamaitice,
Coláiste na hOllscoile,
Baile Átha Cliath.

BOUNDARY BEHAVIOUR OF HOLOMORPHIC AND HARMONIC FUNCTIONS*

Stephen J. Gardiner

Abstract: We give below a survey of some recent results concerning the boundary behaviour of holomorphic and harmonic functions. The unifying theme is the role played by the integral condition

$$\int_{-1}^{1} \frac{\phi(t)}{t^2} dt < \infty, \qquad (1)$$

where ϕ is a non-negative Lipschitz function.

1. Thin sets

Let Ω be a domain (non-empty, connected open set) in the complex plane $\mathbb{C}$. Recall that a function $u : \Omega \to (-\infty, \infty]$, where $u \not\equiv \infty$, is called *superharmonic* on Ω if u is lower semicontinuous, i.e.,

$$u(z_0) \leq \liminf_{z \to z_0} u(z) \qquad (z_0 \in \Omega), \qquad (2)$$

and if

$$u(z_0) \geq \int_0^{2\pi} u(z_0 + re^{i\theta}) \frac{d\theta}{2\pi} \text{ when } \{z : |z - z_0| \leq r\} \subset \Omega. \qquad (3)$$

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