(13), then we define

$$\begin{aligned} G_1 &= B_1^{-1}C_1, \quad H_{1,0} = B_1^{-1}D_1, \\ G_i &= (B_i - A_iG_{i-1})^{-1}C_i, \quad i = 2,3,\dots,N-1 \\ H_{i,0} &= (B_i - A_iG_{i-1})^{-1}(D_i - A_iH_{i-1,0}), \quad i = 2,3,\dots,N. \end{aligned} \tag{14}$$

Then, for $i = 3,4,\dots,n-1$, $k = i-2, i-3, \dots, 1$,

$$H_{k,i-k-1} = H_{k,i-k-2} + (-1)^{i-k-1} \prod_{j=k}^{j=i-2} G_jH_{i-1,0},$$

and the vector components X_i of the solution are given by

$$X_i = H_{i,n-i}, \quad i = 1,2,3,\dots,n.$$

Parallel Algorithms

More recently the problem of designing algorithms which can be efficiently run on several processors working in parallel has attracted considerable interest. Algorithms which are ideal on a single processor may be highly inefficient, or even fail entirely on parallel processors and the design of suitable parallel algorithms for even the commonest problems is a matter for present day research.

Conclusions

The power of computers has given us the following opportunities:

i) to make new discoveries in Mathematics;
ii) in the teaching of Mathematics itself;
iii) to develop new methods (algorithms) which are efficient on computers for the solution of a wide range of problems and particularly so on parallel computers.

D. J. Evans,
Parallel Algorithms Research Centre,
University of Technology,
Loughborough,
Leicestershire LE11 3TU,
England.

CAUCHY'S MATRIX, THE VANDERMONDE MATRIX AND POLYNOMIAL INTERPRETATION

R. Gow

Let K be a field and let $\alpha_1, \dots, \alpha_n$ be elements of K. The $n \times n$ matrix $V = V(\alpha_1, \dots, \alpha_n)$, where

$$V = \begin{pmatrix} 1 & 1 & \dots & 1 \\ \alpha_1 & \alpha_2 & \dots & \alpha_n \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_1^{n-1} & \alpha_2^{n-1} & \dots & \alpha_n^{n-1} \end{pmatrix},$$

is called a Vandermonde matrix. It is an example of a type of matrix known as an *alternant*. See, for example, Chapter XI of [5]. The Vandermonde matrix plays an important role in problems concerning polynomials, symmetric polynomials in particular. The determinant of V is well known to be the difference product

$$\prod_{i>j}(\alpha_i - \alpha_j)$$

and thus V is invertible precisely when the α_i are all different. A proof that $\det V$ has the form stated above may be given as follows. Row operations show that $\det V$ equals the determinant of the $n \times n$ matrix obtained from V by replacing its last row by the row

$$(\, f(\alpha_1) \quad f(\alpha_2) \quad \dots \quad f(\alpha_n)\,),$$

where f is any monic polynomial in $K[x]$ of degree $n-1$. We choose f to equal

$$(x-\alpha_1)\dots(x-\alpha_{n-1}).$$

Then we have $f(\alpha_i) = 0$ for $i \neq n$ and

$$f(\alpha_n) = (\alpha_n - \alpha_1) \dots (\alpha_n - \alpha_{n-1}).$$

If W is the $n \times n$ matrix obtained from V for this choice of f, we easily see that

$$\det V = \det W = f(\alpha_n) \det V(\alpha_1, \dots, \alpha_{n-1})$$

and the result follows easily by induction. Occasionally, evaluations of det V in the literature seem to be unnecessarily complicated, as they refer to facts about homogeneous polynomials. The original evaluation of the determinant is due to Cauchy (Journal de L'École Polytechnique, XVII, 1815).

Let $\mathcal{P}$ denote the n–dimensional vector subspace of $K[x]$ consisting of all polynomials of degree at most $n-1$. The polynomials $1, x, \dots, x^{n-1}$ form the standard basis of $\mathcal{P}$. Let $p_1, \dots, p_n$ be n polynomials in $\mathcal{P}$. Then we may write

$$p_i = \sum_{k=1}^{n} a_{ik} x^{k-1},$$

where the a_{ik} are elements of K. If we evaluate the p_i at the points $\alpha_1, \dots, \alpha_n$, we obtain the matrix relation

$$P = AV,$$

where P is the $n \times n$ matrix whose (i,j) entry is $p_i(\alpha_j)$. Suppose that the α_i are all different, so that V is invertible. We choose the p_i to be the Lagrange interpolation polynomials for the points $\alpha_1, \dots, \alpha_n$, which are defined by the formulae

$$p_i = \frac{p}{p'(\alpha_i)(x - \alpha_i)} \text{ where } p = (x - \alpha_1) \dots (x - \alpha_n)$$

for $1 \leq i \leq n$. Then we find that $p_i(\alpha_i) = 1$ and $p_i(\alpha_j) = 0$ if $i \neq j$. Thus the matrix relation above becomes

$$I_n = AV$$

and the matrix A is the inverse of the Vandermonde matrix. We see that the coefficients of the interpolation polynomials enable us to find the inverse of V.

Suppose now that we have n additional different elements $\beta_1, \dots, \beta_n$ with $\alpha_i \neq \beta_j$ for all i and j. The $n \times n$ matrix

$$\begin{pmatrix} \frac{1}{\alpha_1-\beta_1} & \frac{1}{\alpha_1-\beta_2} & \cdots & \frac{1}{\alpha_1-\beta_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\alpha_n-\beta_1} & \frac{1}{\alpha_n-\beta_2} & \cdots & \frac{1}{\alpha_n-\beta_n} \end{pmatrix}$$

is called a Cauchy matrix. The Cauchy matrix is an example of a *bialternant* or *double alternant*, as discussed in Chapter X1 of [5]. It was introduced by Cauchy in a work [1, pp 151–159] published in 1841, where its determinant is calculated. The Cauchy matrix also appears briefly in Frobenius's development of the irreducible characters of the symmetric group [3, p.153]. In this connection, see also, for example, exercise 6, p.38, of [4]. We shall denote this Cauchy matrix, whose (i,j) entry is $(\alpha_i - \beta_j)^{-1}$, by $C(\alpha, \beta)$. The author has been intrigued with the problem of finding a suitable setting for the Cauchy matrix, analogous to the role of the Vandermonde matrix in polynomial theory. The purpose of this paper is to relate $C(\alpha, \beta)$ to the Vandermonde matrix and show how its determinant and inverse may be evaluated. Since starting this work, we have found that our formula for the inverse is given, in an older formulation, in Section 353 of [5]. The referee of this paper has also pointed out that M. J. Newell has given an approach to the Cauchy matrix on p. 347 of [6] that is rather similar to our presentation in this paper. Thus our findings are certainly not new, but we hope that this subject may be of interest to those who are not specialists in symmetric functions.

We continue to use the interpolation polynomials p_i, based on the points $\alpha_1, \dots, \alpha_n$, and introduce a corresponding family of interpolation polynomials q_i, based on the points $\beta_1, \dots, \beta_n$. Thus

$$q_i = \frac{q}{q'(\alpha_i)(x - \beta_i)} \text{ where } q = (x - \beta_1) \dots (x - \beta_n).$$

Since $\{q_1, \ldots, q_n\}$ is a basis for $\mathcal{P}$, there exist elements e_{ik} of K with

$$p_i = \sum_{k=1}^{n} e_{ik} q_k$$

for $1 \leq i \leq n$. Evaluating the polynomials on each side of the equation above at β_j, we obtain $e_{ij} = p_i(\beta_j)$. Recalling the definition of p_i, we see that

$$e_{ij} = \frac{-p(\beta_j)}{p'(\alpha_i)(\alpha_i - \beta_j)}.$$

Consequently, if $E = (e_{ij})$, we have the relation

$$D(\alpha_1, \ldots, \alpha_n)E = -C(\alpha, \beta)P(\beta_1, \ldots, \beta_n),$$

where $D(\alpha_1, \ldots, \alpha_n)$ and $P(\beta_1, \ldots, \beta_n)$ are the diagonal matrices whose diagonal entries are $p'(\alpha_1)$, ..., $p'(\alpha_n)$ and $p(\beta_1)$, ..., $p(\beta_n)$, respectively. Expressing the polynomials p_i and q_i in terms of powers of x, we have, say,

$$p_i = \sum_{k=1}^{n} a_{ik} x^{k-1}$$

and

$$q_i = \sum_{k=1}^{n} b_{ik} x^{k-1}$$

for $1 \leq i \leq n$. Our discussion earlier shows that if $A = (a_{ij})$ and $B = (b_{ij})$, then

$$A = V(\alpha_1, \ldots, \alpha_n)^{-1}, \quad B = V(\beta_1, \ldots, \beta_n)^{-1}.$$

However, we clearly have $A = EB$ and we obtain the relation

$$V(\alpha)^{-1} = -D(\alpha_1, \ldots, \alpha_n)^{-1}C(\alpha, \beta)P(\beta_1, \ldots, \beta_n)V(\beta)^{-1},$$

where we have written $V(\alpha)$ and $V(\beta)$ in place of $V(\alpha_1, \ldots, \alpha_n)$ and $V(\beta_1, \ldots, \beta_n)$. Thus we have proved the following result.

Theorem 1. *Let α_1, ..., α_n be n different elements in K and let p be the polynomial*

$$(x - \alpha_1)\ldots(x - \alpha_n)$$

in $K[x]$. Let β_1, ..., β_n be a further n different elements in K with $\alpha_i \neq \beta_j$ for all i and j. Let $C(\alpha, \beta)$ be the $n \times n$ Cauchy matrix whose (i, j) entry is $(\alpha_i - \beta_j)^{-1}$. Then we have the equation

$$C(\alpha, \beta) = -D(\alpha_1, \ldots, \alpha_n)V(\alpha)^{-1}V(\beta)P(\beta_1, \ldots, \beta_n)^{-1}.$$

Here $V(\alpha) = V(\alpha_1, \ldots, \alpha_n)$ and $V(\beta) = V(\beta_1, \ldots, \beta_n)$ are the Vandermonde matrices based on the α_i and β_j, respectively, and $D(\alpha_1, \ldots, \alpha_n)$ and $P(\beta_1, \ldots, \beta_n)$ are the $n \times n$ diagonal matrices whose diagonal entries are $p'(\alpha_1)$, ..., $p'(\alpha_n)$ and $p(\beta_1)$, ..., $p(\beta_n)$, respectively.

Corollary 1. *The determinant of the Cauchy matrix is*

$$(-1)^{n(n-1)/2}\frac{\prod_{i>j}(\alpha_i - \alpha_j)\prod_{i>j}(\beta_i - \beta_j)}{\prod_{i,j}(\alpha_i - \beta_j)}.$$

Proof. We may assume that the α_i and the β_j are all different, since otherwise the determinant is clearly 0 and the formula holds in this case. Theorem 1 shows that we have

$$\det C(\alpha, \beta) = (-1)^n \frac{\prod_{i=1}^{n} p'(\alpha_i)\prod_{i>j}(\beta_i - \beta_j)}{\prod_{i=1}^{n} p(\beta_i)\prod_{i>j}(\alpha_i - \alpha_j)}.$$

However, it is easy to verify that

$$\prod_{i=1}^{n} p'(\alpha_i) = (-1)^{n(n-1)/2}\prod_{i>j}(\alpha_i - \alpha_j)^2$$

and

$$\prod_{i=1}^{n} p(\beta_i) = (-1)^{n^2}\prod_{i,j}(\alpha_i - \beta_j)$$

and the result follows.

It is easy to find the inverse of the Cauchy matrix in the case that its determinant is non-zero. In Theorem 1, we write $V(\alpha)$, $V(\beta)$, $D(\alpha)$ and $P(\beta)$ in place of $V(\alpha_1, \ldots, \alpha_n)$, $V(\beta_1, \ldots, \beta_n)$, $D(\alpha_1, \ldots, \alpha_n)$ and $P(\beta_1, \ldots, \beta_n)$, respectively. Then Theorem 1 gives

$$C(\alpha, \beta) = -D(\alpha)V(\alpha)^{-1}V(\beta)P(\beta)^{-1}.$$

Interchanging the roles of the α_i and β_j, we have

$$C(\beta, \alpha) = -E(\beta)V(\beta)^{-1}V(\alpha)Q(\alpha)^{-1},$$

where $E(\beta)$ and $Q(\alpha)$ are the $n \times n$ diagonal matrices whose i-th diagonal entries are $q'(\beta_i)$ and $q(\alpha_i)$ for $1 \leq i \leq n$, respectively. Assuming that $C(\alpha, \beta)$ is invertible, we obtain the relation

$$C(\alpha, \beta)^{-1} = P(\beta)E(\beta)^{-1}C(\beta, \alpha)Q(\alpha)D(\alpha)^{-1}.$$

We also observe that $C(\beta, \alpha) = -C(\alpha, \beta)'$, the prime denoting transpose. We have therefore proved the following result.

Theorem 2. *Let $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_n$ be $2n$ different elements in K and let p and q be the polynomials*

$$(x - \alpha_1) \ldots (x - \alpha_n) \text{ and } (x - \beta_1) \ldots (x - \beta_n),$$

respectively. Then we have the relation

$$C(\alpha, \beta)^{-1} = -\Lambda_1 C(\alpha, \beta)' \Lambda_2,$$

where Λ_1 and Λ_2 are the diagonal matrices whose i-th diagonal entries are $p(\beta_i)/q'(\beta_i)$ and $q(\alpha_i)/p'(\alpha_i)$, respectively. In particular, the (i, j) entry of $C(\alpha, \beta)^{-1}$ is

$$\frac{p(\beta_i)q(\alpha_j)}{(\beta_i - \alpha_j)p'(\alpha_j)q'(\beta_i)}.$$

As an example of the use of this formula, we consider the case that $\alpha_i = i - 1$ and $\beta_i = -i$ for $1 \leq i \leq n$. The corresponding

Cauchy matrix based on these values is

$$\begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \cdots \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots \\ \vdots & \vdots & \vdots & \ddots \\ \frac{1}{n} & \frac{1}{n+1} & \frac{1}{n+2} & \cdots \end{pmatrix}.$$

This matrix is usually called a Hilbert matrix. The polynomials p and q for this matrix are

$$x(x-1)\ldots(x-n+1) \text{ and } (x+1)(x+2)\ldots(x+n).$$

Theorem 2 shows that the (i, j) entry of the inverse of the Hilbert matrix is

$$\frac{(-1)^{i+j}(n+i-1)!(n+j-1)!}{(i+j-1)(i-1)!^2(j-1)!^2(n-i)!(n-j)!},$$

which equals

$$(-1)^{i+j}(i+j-1)\binom{n+i-1}{n-j}\binom{n+j-1}{n-i}\binom{i+j-2}{i-1}^2,$$

as shown in [2, p.306].

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R. Gow,
Department of Mathematics,
University College,
Belfield,
Dublin 4.

SOME QUESTIONS CONCERNING THE VALENCE OF ANALYTIC FUNCTIONS

J. B. Twomey

In this short note we discuss, and illustrate by means of some examples, certain questions concerning the *valence* of analytic functions of one complex variable, that is, the number of times such functions take their values. We present a theorem which asserts the existence of certain constants relating to the valence of analytic functions in the unit disc, and conclude the note by raising some questions regarding these constants for the reader.

We begin with a definition. Suppose a function f is analytic in a domain D in the complex plane. We say that f is p-*valent* in D, p a positive integer, if (*i*) f takes no value more than p times in D, and (*ii*) f takes at least one value exactly p times in D. If $p = 1$ we have, of course, a *univalent* (or one-to-one) function. The following result for univalent functions is elementary and known:
(1) *If f is analytic in the unit disc $U = \{z : |z| < 1\}$ and univalent in the annulus*

$$A(\delta) \equiv \{z : \delta < |z| < 1\},$$

where $0 < \delta < 1$, then f is univalent in the full disc U.
This result is an easy consequence of Darboux's theorem [1, p. 115]: If f is analytic on and inside a simple closed curve γ, and f takes no value more than once on γ, then f is univalent inside γ.

It is natural to attempt to generalize (1) and to ask whether there is an analogous result for p-valent functions when $p > 1$. (This question was first posed by A. W. Goodman in a seminar in Tampa many years ago and this author's interest in these problems dates — albeit discontinuously — from that occasion.) We note immediately that the direct analogue of (1), namely