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Partially Ordered Groups

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1. Introduction

Ordered algebraic structures, such as ordered fields and ordered vector spaces, have long been studied in mathematics, both for their own intrinsic interest, and for their applications. In recent years partially ordered groups have become important in connection with the theory of operator algebras, particularly C^*-algebras. The most important way in which a connection is manifested is by means of K-theory. For example, if A is an AF-algebra, then $K_0(A)$ is a partially ordered group, and this group can be used to analyse and classify A. We discuss this in Section 3. In another direction, if a partially ordered group is given, one can associate to it a certain universal C^*-algebra. In the good cases, this algebra turns out to be the C^*-algebra generated by the Toeplitz operators with continuous symbols on the dual group. The theory of these algebras and operators has been developed by the author and by others, and we discuss some of its aspects both in the following section and in Section 3.

A *partially ordered group* is a pair $(G, \leq)$ consisting of a discrete abelian group G, and a partial order $\leq$ on G which is translation-invariant, that is, if $x \leq y$, then $x + z \leq y + z$ $(x, y, z \in G)$, and the positive cone $G^+ = \{x \in G \mid 0 \leq x\}$ generates G (that is, $G = G^+ - G^+$).

Although this definition is a purely algebraic one, we observe that the theory of partially ordered groups has been strongly influenced by functional analysis—specifically, by the theory of partially ordered vector spaces. Also, as indicated above, the applications to operator algebras have largely determined the direction of recent research in this area.

If G is an abelian group, a *cone* of G is a subset C such that $C + C \subseteq C$, $C \cap (-C) = \{0\}$, and $G = C - C$. Given a cone C, one can define a partial order on G by setting $x \leq y$ if $y - x \in C$. This partial order is the unique one making G a partially ordered group whose positive cone is C.

Clearly, if G is a partially ordered group, then G^+ is a cone of G.

If G is a subgroup of $\mathbf{R}$, it is a partially ordered group, with positive cone $G^+ = G \cap \mathbf{R}^+$. We shall always understand the order on subgroups of $\mathbf{R}$ to be this one. The group $\mathbf{Z}^n$ is a partially ordered group, where the positive cone is $\mathbf{N}^n$. Such a group, and any partially ordered group isomorphic to it (as a partially ordered group), is called a *simplicial* group.

A large class of examples of partially ordered groups is obtained by considering the self-adjoint part of a C^*-algebra. Since these algebras feature prominently in the sequel, we recall their definition. A *C^*-algebra* is a Banach algebra endowed with an involution operation $a \mapsto a^*$ such that $\|a^*a\| = \|a\|^2$ for all elements a. Every such algebra has a faithful representation as a norm-closed self-adjoint algebra of operators on a Hilbert space. This class of algebras has a very well developed theory, and a vast range of important applications to other areas of mathematics, such as algebraic topology and differential geometry, and to the sciences, in particular, to quantum mechanics. For an introduction to C^*-algebras, see [9]. If A_{sa} is the set of hermitian elements ($a^* = a$) of a C^*-algebra A, then it is a partially ordered group, where the positive cone is the set of all squares a^2 ($a \in A_{sa}$).

An important way in which partially ordered groups arise naturally is given by the process of deriving a group from a semigroup, the Grothendieck construction. Let C be an abelian cancellative semigroup with zero element. An equivalence relation is defined on $C \times C$ by setting $(x, y) \sim (x', y')$, when $x + y' = x' + y$. If $[x, y]$ denotes the equivalence class of (x, y), and G is the set of equivalence classes, then G is an abelian group under the addition operation $[x, y] + [x', y'] = [x + x', y + y']$. The zero is $[0, 0] = [x, x]$, and the additive inverse of $[x, y]$ is $[y, x]$. The semigroup C can be embedded in G by means of the injective homomorphism $x \mapsto [x, 0]$, and then $G = C - C$. The group G is the *enveloping* group of C, and has the universal property that every homomorphism from C into an abelian group extends uniquely to a homomorphism of G into the group. The prototypical example, of course, is given by $C = \mathbf{N}$, and $G = \mathbf{Z}$. The Grothendieck construction is important in a number of situations, as for instance in the K_0-theory of unital C^*-algebras. Partial order comes into this because, although there are advantages in replacing a semigroup by its enveloping group, in some cases we also need to keep the original semigroup in mind as well (an example is given below, in connection with stable isomorphism of AF-algebras). If C has the property that $x + y = 0$ implies that $x = y = 0$ ($x, y \in C$), then C is a cone of G. Thus, in this case, G is a partially ordered group in a natural way. In this fashion, $K_0(A)$ is a partially ordered group, if A is a unital AF-algebra. (Note that $K_0(A)$ is *not* always a partially ordered group for arbitrary C^*-algebras.)

We shall give more examples as we proceed.

Amongst partially ordered groups, three subclasses are particularly important, namely, archimedean groups, totally ordered groups, and dimension groups. In the following section we shall confine our discussion to totally ordered groups and archimedean groups. We defer discussion of the much larger class of dimension groups to Section 3.

2. Ordered groups

A *(totally) ordered group* is a partially ordered group $(G, \leq)$ in which every pair of elements is comparable, that is, for all $x, y \in G$, either $x \leq y$ or $y \leq x$. Of course, the subgroups of $\mathbf{R}$ are ordered groups, but the simplicial group $\mathbf{Z}^n$ is not, unless $n = 1$.

If G and H are ordered groups, we can endow the product group $G \times H$ with a natural order making it an ordered group. Define $(x, y) \leq (x', y')$ to mean that either $x < x'$, or $x = x'$ and $y \leq y'$. This order is called the *lexicographic* order, and when endowed with it, $G \times H$ is called the *lexicographic* product. In a similar manner, one can define the lexicographic product of ordered groups $G_1, \ldots, G_n$, or indeed, of an infinite sequence $(G_n)_{n=1}^{\infty}$ of ordered groups.

The same group may admit different total orders. For instance, $\mathbf{Z}^2$ has the lexicographic order, and also, if θ is an irrational number, it has another order, namely the one whose cone is the set of all elements (m, n) such that $m + \theta n \geq 0$.

Note that an ordered group is necessarily torsion free. Also, it is not hard to show that a torsion-free partially ordered group G is an ordered group if and only if G^+ is a maximal cone. In fact, any torsion-free abelian group can be made into an ordered group, and we are therefore assured of a large supply of examples of ordered groups. (This result is due to Levi [7].)

Let G be a discrete abelian group, and denote by $\hat{G}$ its Pontryagin dual group. If x is a non-zero element of G of finite order, the set $\{\gamma(x) \mid \gamma \in \hat{G}\}$ is finite, and not a singleton, so it is disconnected. Hence, $\hat{G}$ is disconnected. Conversely, if $\hat{G}$ is disconnected, one can show that G admits a non-zero element of finite order ([13], p47). Thus, G is torsion free if and only if $\hat{G}$ is connected. This turns out to be important in the theory of Toeplitz operators defined relative to ordered groups, which we now discuss briefly.

If G is an ordered group, the *Hardy space* $H^2(G)$ is the L^2 norm closed linear subspace of $L^2(\hat{G})$ consisting of all functions f whose Fourier transform $\hat{f}$ is supported in G^+. If $G = \mathbf{Z}$, one gets the classical Hardy space on the circle group $\mathbf{T}$. Much of the original H^p space theory has been extended to the more general situation by Helson and Lowdenslager—for a detailed account see Rudin [13].

Let P denote the orthogonal projection of $L^2(\hat{G})$ onto $H^2(G)$. If φ is a complex-valued continuous function on $\hat{G}$, then the bounded linear operator

$$H^2(G) \to H^2(G), \quad f \mapsto P(\varphi f),$$

is denoted by T_φ, and called the *Toeplitz* operator with *symbol* φ. Using the fact that $\hat{G}$ is connected, the author gave a new proof of a result of Widom and Devinatz which asserts that the spectrum of T_φ is connected [10]. An important question concerning T_φ is its invertibility. In the classical case ($G = \mathbf{Z}$) it is well known that invertibility of T_φ is equivalent to the existence of a continuous logarithm for φ. The author extended this result to the general case [10], and here again the proof uses the connectivity of $\hat{G}$.

Denote by $\mathbf{A}(G)$ the C^*-algebra of operators on $H^2(G)$ generated by the Toeplitz operators. This algebra, and certain of its subalgebras, turn out to be interesting from the point of view of C^*-algebra theory. For instance, one gets a new class of simple C*-subalgebras (a C^*-algebra is *simple* if it has no non-trivial closed ideals—it is important to have examples of such algebras, but they are not always easy to obtain). The K-groups of these algebras are difficult to compute and have received much attention recently. For subgroups of the reals, the K-theory has been completely computed, and for general ordered groups some important partial information has been obtained. We shall return to this topic later, in connection with dimension groups.

An *archimedean* group is an ordered group G such that for every pair $x, y > 0$ there exists a positive integer n such that $x \leq ny$. The subgroups of $\mathbf{R}$ are clearly archimedean, and in fact these are all the archimedean groups, up to ordered group isomorphism (for a proof see [13]). If G is an archimedean group, then the commutator ideal of $\mathbf{A}(G)$ is simple, a result due to Douglas [3]. (The *commutator* ideal is the smallest closed ideal containing all of the additive commutators $ab - ba$.) The author showed that the converse is also true—if $\mathbf{A}(G)$ has simple commutator ideal, then G is archimedean. Thus, order properties of the group are strongly reflected in algebraic properties of its associated C^*-algebra, and conversely.

In analysing the C^*-algebras $\mathbf{A}(G)$, the author discovered the following result concerning ordered groups, which may be new: *A finitely-generated ordered group is a lexicographic product of a finite number of archimedean groups.* This does *not*, by any means, reduce the study of the algebras $\mathbf{A}(G)$ to the case where G is a subgroup of $\mathbf{R}$, but it is a useful result in the theory of these algebras (see [11]).

3. Dimension groups

The groups of the title of this section form a class of partially ordered groups which arise in the study of certain C^*-algebras, namely AF-algebras. They have been the subject of intensive study, and now have a fairly well-developed theory. For a comprehensive treatment, see Goodearl's recent AMS monograph [6]. Dimension groups are also covered in [1], [4] and [5].

An *AF-algebra* is a C^*-algebra A having an increasing sequence of finite-dimensional C^*-subalgebras A_n whose union $\cup_n A_n$ is dense in A. An example of such an algebra is the set of all compact operators on a separable Hilbert space. On the other hand, the C^*-algebra of all bounded operators is not an AF-algebra, unless the Hilbert space is finite-dimensional. The class of AF-algebras is sufficiently close to that of the finite-dimensional C^*-algebras to be tractable, but it is nevertheless a highly non-trivial class and exhibits typical C^*-algebra behaviour. Some C^*-algebras which are important in the theory of quantum mechanics belong to this class.

For the sake of simplicity, we shall only consider unital AF-algebras.

If A is a finite-dimensional C^*-algebra, it is easy to see that for some n, its K-group $K_0(A)$ is equal to a simplicial group $\mathbf{Z}^n$. If now A is assumed to be an AF-algebra, then by definition, it is a direct limit of finite-dimensional C^*-algebras, and therefore by continuity of the functor K_0, the partially ordered group $K_0(A)$ is the direct limit of a sequence of simplicial groups. These groups, direct limits of simplicial groups, are called *dimension* groups. (The positive cone of a K_0-group is thought of as the set of "dimensions" of the projections of the algebra, and of its matrix algebras.)

It is a remarkable, and very important, result of this theory that dimension groups have been given a very nice abstract and *usable* characterization:

Theorem 3.1 *A countable partially ordered group G is a dimension group if and only if the following conditions are satisfied:*

(1) If $nx \geq 0$ and $n > 0$, then $x \geq 0$;
(2) If $x_i \leq y_j$ for $i, j = 1, 2$, then there exists $z \in G$ such that $x_i \leq z \leq y_j$.

Condition (2) is called the *Riesz interpolation property*. The theorem is due to E. Effros, D. Handelman, and C.–L. Shen. For a proof see [5].

A dimension group closely reflects the structure of its corresponding AF-algebra. For instance, the closed ideals of the algebra correspond to certain subgroups of the dimension group, called its "ideals." Thus, if the dimension group is *simple*, that is, has no non-trivial ideals, the AF-algebra is simple.

If one wants to construct an AF-algebra with certain unusual properties, one may be able to do this by interpreting the properties in terms of the dimension group, and trying to construct the latter. In an important instance where this approach has been taken, and has paid off very well, B. Blackadar obtained a certain AF-algebra with unusual properties, from which he in turn constructed a C^*-algebra which is simple, yet has no non-trivial projections (self-adjoint idempotent elements). This solved a difficult problem which had been open for many years.

Since simple dimension groups are particularly important, we give some examples to illustrate the possibilities.

Every countable subgroup of $\mathbf{R}$ is a simple dimension group.

Let $G = \mathbf{Q}^n$, and define the positive cone to be

$$G^+ = \{(x_1, \ldots, x_n) \mid x_1, \ldots, x_n > 0\} \cup \{(0, \ldots, 0)\}.$$

The corresponding partial order on G is called the *strict* order. It is easy to see that G is a simple dimension group.

If $G = \mathbf{Q}^2$, where the positive cone is $G^+ = \{(x, y) \mid x > 0\} \cup \{(0, 0)\}$, then G is a simple dimension group.

If A and B are AF-algebras, under what conditions on their dimension groups are they isomorphic? The answer, due to G. Elliott, is easy to state, but the proof is difficult. A necessary and sufficient condition for A and B to be isomorphic is that there is an order isomorphism of the corresponding dimension groups, which is *unital* in the sense that the K_0-classes of the units of A and B correspond. If the dimension groups are only order isomorphic (with no assumption that the isomorphism is unital), then the algebras are *stably isomorphic*, which may be loosely asserted to mean they have the same representation theory.

We finish up by returning briefly to the theory of Toeplitz operators. Let G be an ordered group. If $\varphi = (\varphi_{ij})$ is a square matrix of order n whose entries are continuous complex-valued functions on $\hat{G}$, define the Toeplitz operator T_φ to be the matrix $(T_{\varphi_{ij}})$ (as an operator this acts on the orthogonal direct sum of n copies of $H^2(G)$). It is shown by the author in [12] that if T_φ is invertible, then φ is invertible and its class (its "topological index") in the K-group $K_1(C(\hat{G}))$ is the zero element. This extends a result known for $\mathbf{Z}$, and more generally, for subgroups of $\mathbf{R}$. The proof involves K-theoretic computations which use the fact that ordered groups are dimension groups, and therefore may be written as direct limits of simplicial groups. Actually, rather more is proved, and the interested reader is referred to [12] for details.

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