IRISH MATHEMATICAL SOCIETY

BULLETIN

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The aim of the Bulletin is to inform Society members about the activities of the Society and about items of general mathematical interest. It appears twice each year, in March and December. The Bulletin is supplied free of charge to members; it is sent abroad by surface mail. Libraries may subscribe to the Bulletin for IR£20.00 per annum.

The Bulletin seeks articles of mathematical interest written in an expository style. All areas of mathematics are welcome, pure and applied, old and new. The Bulletin is typeset using TEX. Authors are invited to submit their articles in the form of TEX input files if possible, in order to ensure speedier processing.

Correspondence concerning the Bulletin should be addressed to:

Irish Mathematical Society Bulletin
Department of Mathematics
University College
Galway
Ireland

Correspondence concerning the Problem Page should be sent directly to the Problem Page Editor at the following address:

Faculty of Mathematics
Open University
Milton Keynes, MK7 6AA
UK

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NOTES ON APPLYING FOR I.M.S. MEMBERSHIP

1. The Irish Mathematical Society has reciprocity agreements with the American Mathematical Society and the Irish Mathematics Teachers Association.

2. The current subscription fees are given below.

Institutional member	IR£50.00
Ordinary member	IR£10.00
Student member	IR£4.00
I.M.T.A. reciprocity member	IR£5.00

The subscription fees listed above should be paid in Irish pounds (puint) by means of a cheque drawn on a bank in the Irish Republic, a Eurocheque, or an international money-order.

3. The subscription fee for ordinary membership can also be paid in a currency other than Irish pounds using a cheque drawn on a foreign bank according to the following schedule:

   If paid in United States currency then the subscription fee is US$18.00.
   If paid in sterling then the subscription fee is £10.00 stg.
   If paid in any other currency then the subscription fee is the amount in that currency equivalent to US$18.00.

   The amounts given in the table above have been set for the current year to allow for bank charges and possible changes in exchange rates.

4. Any member with a bank account in the Irish Republic may pay his or her subscription by a bank standing order using the form supplied by the Society.

5. The subscription fee for reciprocity membership by members of the American Mathematical Society is US$10.00.

6. Subscriptions normally fall due on 1 February each year.

7. Cheques should be made payable to the Irish Mathematical Society. If a Eurocheque is used then the card number should be written on the back of the cheque.

8. Any application for membership must be presented to the Committee of the I.M.S. before it can be accepted. This Committee meets twice each year.

9. Please send the completed application form with one year's subscription fee to

   The Treasurer, I.M.S.
   Department of Mathematics
   University College
   Dublin
   Ireland

EDITORIAL

The 1991 issues of the Bulletin did not appear until 1992; we apologize for this, and we assure readers that we intend to get back on schedule as soon as we can.

We therefore invite the submission of articles for publication. As is stated inside the front cover, we seek articles of general mathematical interest, written in expository style. We particularly welcome surveys of areas in pure or applied mathematics, notes presenting viewpoints or proofs of interest to those teaching mathematics at undergraduate or postgraduate level, articles concerned with mathematical education at all levels, and articles on any aspect of the history of mathematics. We also invite the submission of research announcements and we expect that the abstracts of Irish Ph.D. theses will be of interest to readers.

The jobs of the editor and associate editor have been made much easier by the appointment of a production manager. Those contributors who are able to do so can make the production manager's job easier by submitting articles in TEX. They (and others) will be interested in the article in issue number 27 by the present production manager, Mícheál Ó Searcóid, in which he describes two files; the first is a format file enabling contributors to see what their articles will look like in the Bulletin and the second is a macro file which can be used to compile lists of references for papers appearing in this and other journals. Information on how to obtain copies of these packages is given in the instructions to authors inside the back cover.

CONSTITUTION OF THE IRISH MATHEMATICAL SOCIETY

as amended by the Ordinary Meeting
held on 21 December 1984

1. The Irish Mathematical Society shall consist of Ordinary and Honorary Members.

2. Any person may apply to the Treasurer for membership by paying one year's membership fee. His admission to membership must then be confirmed by the Committee of the Society. Candidates for honorary membership may be nominated by the Committee only, following a proposal of at least three members of the Society. Nominations for honorary membership must be made at one Ordinary Meeting of the Society and voted upon at the next, a simple majority of the members present being necessary for election.

3. Every Ordinary member shall pay subscription to the funds of the Society at the times and of the amounts specified in the Rules.

4. The Office-Bearers shall consist of a President, a Vice-President, a Secretary, a Treasurer. The Office of President or Vice-President may be held in conjunction with any of the other offices.

5. The Committee shall consist of the President, the Vice-President, the Secretary, the Treasurer, and eight additional members. No person shall serve as an additional member for more than three terms consecutively.

6. There shall normally be at least two Ordinary Meetings in a session.

7. Notice of a motion to repeal or alter part of the Constitution shall be given at one Ordinary Meeting. Written notice of one month shall be given to all members before the next Ordinary Meeting at which the motion shall be voted upon, being carried if it receives the consent of two-thirds of the members present.

8. One month's written notice of a motion to repeal or alter a Rule, or to enact a new Rule, shall be given to all members before the Meeting at which it is to be voted upon, the motion being carried if it receives the consent of a simple majority of the members present.

9. All questions not otherwise provided for in the Constitution and Rules shall be decided by a simple majority of members present at a Meeting. Eleven Ordinary members shall form a quorum for such business.

RULES

as amended by the Ordinary Meeting
held on 16 April 1992

These rules shall be subject to the over-riding authority of the Constitution.

SUBSCRIPTIONS:

1. Every Ordinary Member shall pay, on election to membership and during January in each succeeding session, an annual subscription to be determined by the Committee. A change in the annual subscription shall be ratified by a Meeting of the Society.

2. Ordinary Members whose subscriptions are more than eighteen months in arrears shall be deemed to have resigned from the Society.

OFFICERS AND COMMITTEE

3. The election of the Office-Bearers and the additional members of the Committee shall take place at the last Ordinary Meeting of each session.
4. The term of office of the Office-Bearers and the Committee shall be two sessions starting on the first day of January that follows the Ordinary Meeting at which they were elected. The President and the Vice-President may not continue in office for more than two consecutive terms.
5. On alternate years elections for the following positions will take place :
   (a) President, Vice-President, and half of the additional members of the Committee.
   (b) Secretary, Treasurer, and one half of the additional members of the Committee.
6. Each session shall commence on the 1st day of January and last until the following 31st day of December.
7. The Committee shall meet a least twice during each session, the President to be convener. Five shall form a quorum.
8. The Secretary shall keep minutes of the Meetings of the Society and of the Committee and shall issue notice of meetings to members resident in Ireland.
9. A Financial Statement for each session shall be written by the Treasurer holding office in that session, shall be duly audited by two persons appointed by the Committee, and shall be submitted to the First Ordinary Meeting that follows that session.

IMPORTANT MESSAGE TO MEMBERS USING BANK STANDING ORDERS

If you use a bank standing order to pay your subscription fee to the Society then you should contact your bank and make sure that

(i) the current subscription fee of IR£10.00 is being paid,
(ii) payment is made annually and *not* monthly,
(iii) payment is made into the correct bank account,
(iv) any previous order in favour of the Society has been cancelled.

The Society changed its bank early in 1990. The account into which subscription fees should be paid is

The Irish Mathematical Society
Bank of Ireland
U.C.D. Branch, Belfield
Dublin 4

The account number is 17063243; the bank code number is either 90–13–86 or 90–13–51.

Several members are still using a bank standing order completed before the Society changed both its subscription fee and its bank. Not only are these members now at least one subscription fee in arrears but their subscription fees are being paid, in the first instance, into a bank account that has been closed for over two years.

Some banks have made mistakes when processing a current standing order. I cannot correct these mistakes myself since I cannot, of course, give instructions about someone else's bank account. Any member who discovers such a mistake in processing his standing order should ask his bank to correct it. This can always be done without extra cost to either the member or the Society.

I do not wish to discourage members who can do so from paying their subscription fees by means of a standing order; most standing orders are correctly processed and each such one lowers the cost of collecting subscriptions and helps the Society's cash flow. But I do ask any member using a standing order to be careful.

Any member who wants to complete a new standing order to pay his subscription fees should ask me to send him the appropriate form.

David Tipple, Hon. Treasurer

Minutes of Meetings of the Irish Mathematical Society

Ordinary Meeting

20 December 1991

The Irish Mathematical Society held an Ordinary Meeting at 12.15 pm on Friday 20th December 1991 in the DIAS, 10 Burlington Road.

Fifteen members were present. The President, R. Timoney, was in the chair.

1. The **minutes** of the meeting of 28th March 1991 were approved and signed.
2. **Matters arising:** D. O'Donovan's survey of TCD graduates has been published in the Bulletin. R. Timoney urged other institutions to conduct a similar survey.
3. **President's Report:** The main activities for the year were
   - **European Mathematical Society.** The IMS became an institutional member of the EMS this year and IMS members were offered the opportunity of becoming individual members. Only 11 availed of this opportunity, but I hope more will do so for 1992.

     The EMS has produced two issues of its Newsletter (plans are to produce it quarterly from now on) and the most recent issue contained an article on the IMS written by me.

     Plans for the European Congress (to be held in Paris on July 6–10, 1992) are advanced and consideration of the location for 1996 has begun.

     Brendan Goldsmith is our representative for EMS affairs and he has been dealing with things with his usual enthusiasm and efficiency. He is also "Irish correspondent" for the EMS Bulletin and would welcome items for inclusion (his email address is **bgoldsmith@dit.ie**). He would also welcome subscriptions for EMS individual membership in 1992 (cheques for £11 payable to the IMS).

- **RIA.** Brendan Goldsmith has been extremely attentive as IMS representative on the National Committee for Mathematics and he has just been nominated for a further 4 years.
- **Points for Leaving Cert Maths.** This issue was debated at length at the Christmas 1990 and Easter 1991 Ordinary Meeting.
- **September Meeting.** The 1991 meeting was successfully organized at UCG by Ray Ryan and Graham Ellis. The 1992 meeting will be the first at an RTC — Waterford.
- **Bulletin.** Thanks to the efforts of Fergus Gaines, two issues of the Bulletin have been issued in 1991. Unfortunately we have not really succeeded in making significant progress in catching up on the schedule. This must be a priority for 1992. The Committee is trying to install mechanisms which will enable the technical printing side of producing the Bulletin to be separated from the editor's domain. The new editor is James Ward of UCG, but he will be getting significant help from Rex Dark in the short term.

4. **September Meeting:** The next annual mathematical meeting of the Society will be held in Waterford RTC on Thursday 3rd and Friday 4th September 1992.,The Organizers, P. Barry and B. McCann, would welcome suggestions for speakers. It is envisaged that the meeting will have a slightly applied bent.
5. **European Mathematical Society:**
   a) At the 1992 Paris Congress there will be a round-table discussion on mathematical collaboration with developing countries. B. Goldsmith would like details of any such Irish collaboration.
   b) There will also be round-table discussions on: (i) exchange of students and harmonization; (ii) women mathematicians.
   c) R. Timoney announced the results of a survey he carried out for the round-table discussion on women. Only 5% of the permanent mathematical staff of Irish universities are women. None of the nine full professors are women. Last year two women and one man were awarded PhD's. About 30% of honours mathematics students are women. No shift in the proportion of women mathematics students has been detected.

6. **Rule change:** A motion to change the rules of the Society regarding the financial year was deferred to the Easter meeting. This is to allow members three months notice of the proposed change.
7. **Elections:** The following were elected, unopposed, to the committee
   (* denotes re-election):

Committee member	Proposer	Seconder
G. Ellis* (Secretary)	R. O. Watson	T. Laffey
E. Gath	G. Ellis	T. Laffey
D. Hurley	M. Ó Searcóid	R. Timoney
P. Mellon	B. Goldsmith	A. O'Farrell
C. Nash	A. O'Farrell	D. Simms
D. Tipple* (Treasurer)	M. Ó Searcóid	S. Dineen

The following have one more year of office:

R. Timoney (President), B. Goldsmith (Vice-President), F. Gaines, F. Holland, B. McCann, M. Ó Searcóid.

The following have left the committee:

G. Enright, A. O'Farrell, D. Simms, R. O. Watson.

8. **AOB:**
   a) R. Timoney urged individual members of the Society to support a campaign about human rights in South Africa.
   b) S. Smale will talk to TCD Mathematics Society at 8 pm on 13th January in Room 2041b of the TCD Arts Block.
   c) The deadline for applications for Eolas funding of basic research is 31st January 1992. It was felt that the amount of money awarded to any discipline is proportional to the number of applications for research in that discipline.
   d) Members are urged to write/solicit articles for the Bulletin. Articles should preferably be in TEX.

There is to be a Research Announcement section in the Bulletin, along the lines of that in the Bulletin of the AMS. Research announcements must be backed up by a preprint.

R. Dark will act as editor of the Bulletin until October 1992.

M. Ó Searcóid has agreed to help with the typesetting.

Graham Ellis,
University College,
Galway.

Partially Ordered Groups

Gerard J. Murphy

1. Introduction

Ordered algebraic structures, such as ordered fields and ordered vector spaces, have long been studied in mathematics, both for their own intrinsic interest, and for their applications. In recent years partially ordered groups have become important in connection with the theory of operator algebras, particularly C^*-algebras. The most important way in which a connection is manifested is by means of K-theory. For example, if A is an AF-algebra, then $K_0(A)$ is a partially ordered group, and this group can be used to analyse and classify A. We discuss this in Section 3. In another direction, if a partially ordered group is given, one can associate to it a certain universal C^*-algebra. In the good cases, this algebra turns out to be the C^*-algebra generated by the Toeplitz operators with continuous symbols on the dual group. The theory of these algebras and operators has been developed by the author and by others, and we discuss some of its aspects both in the following section and in Section 3.

A *partially ordered group* is a pair $(G, \leq)$ consisting of a discrete abelian group G, and a partial order $\leq$ on G which is translation-invariant, that is, if $x \leq y$, then $x + z \leq y + z$ $(x, y, z \in G)$, and the positive cone $G^+ = \{x \in G \mid 0 \leq x\}$ generates G (that is, $G = G^+ - G^+$).

Although this definition is a purely algebraic one, we observe that the theory of partially ordered groups has been strongly influenced by functional analysis—specifically, by the theory of partially ordered vector spaces. Also, as indicated above, the applications to operator algebras have largely determined the direction of recent research in this area.

If G is an abelian group, a *cone* of G is a subset C such that $C+C \subseteq C$, $C \cap (-C) = \{0\}$, and $G = C - C$. Given a cone C, one can define a partial order on G by setting $x \le y$ if $y - x \in C$. This partial order is the unique one making G a partially ordered group whose positive cone is C.

Clearly, if G is a partially ordered group, then G^+ is a cone of G.

If G is a subgroup of $\mathbf{R}$, it is a partially ordered group, with positive cone $G^+ = G \cap \mathbf{R}^+$. We shall always understand the order on subgroups of $\mathbf{R}$ to be this one. The group $\mathbf{Z}^n$ is a partially ordered group, where the positive cone is $\mathbf{N}^n$. Such a group, and any partially ordered group isomorphic to it (as a partially ordered group), is called a *simplicial* group.

A large class of examples of partially ordered groups is obtained by considering the self-adjoint part of a C^*-algebra. Since these algebras feature prominently in the sequel, we recall their definition. A *C^*-algebra* is a Banach algebra endowed with an involution operation $a \mapsto a^*$ such that $\|a^*a\| = \|a\|^2$ for all elements a. Every such algebra has a faithful representation as a norm-closed self-adjoint algebra of operators on a Hilbert space. This class of algebras has a very well developed theory, and a vast range of important applications to other areas of mathematics, such as algebraic topology and differential geometry, and to the sciences, in particular, to quantum mechanics. For an introduction to C^*-algebras, see [9]. If A_{sa} is the set of hermitian elements ($a^* = a$) of a C^*-algebra A, then it is a partially ordered group, where the positive cone is the set of all squares a^2 ($a \in A_{sa}$).

An important way in which partially ordered groups arise naturally is given by the process of deriving a group from a semigroup, the Grothendieck construction. Let C be an abelian cancellative semigroup with zero element. An equivalence relation is defined on $C \times C$ by setting $(x,y) \sim (x',y')$, when $x + y' = x' + y$. If $[x,y]$ denotes the equivalence class of (x,y), and G is the set of equivalence classes, then G is an abelian group under the addition operation $[x,y] + [x',y'] = [x+x', y+y']$. The zero is $[0,0] = [x,x]$, and the additive inverse of $[x,y]$ is $[y,x]$. The semigroup C can be embedded in G by means of the injective homomorphism $x \mapsto [x,0]$, and then $G = C - C$. The group G is the *enveloping* group of C, and has the universal property that every homomorphism from C into an abelian group extends uniquely to a homomorphism of G into the group. The prototypical example, of course, is given by $C = \mathbf{N}$, and $G = \mathbf{Z}$. The Grothendieck construction is important in a number of situations, as for instance in the K_0-theory of unital C^*-algebras. Partial order comes into this because, although there are advantages in replacing a semigroup by its enveloping group, in some cases we also need to keep the original semigroup in mind as well (an example is given below, in connection with stable isomorphism of AF-algebras). If C has the property that $x + y = 0$ implies that $x = y = 0$ ($x, y \in C$), then C is a cone of G. Thus, in this case, G is a partially ordered group in a natural way. In this fashion, $K_0(A)$ is a partially ordered group, if A is a unital AF-algebra. (Note that $K_0(A)$ is *not* always a partially ordered group for arbitrary C^*-algebras.)

We shall give more examples as we proceed.

Amongst partially ordered groups, three subclasses are particularly important, namely, archimedean groups, totally ordered groups, and dimension groups. In the following section we shall confine our discussion to totally ordered groups and archimedean groups. We defer discussion of the much larger class of dimension groups to Section 3.

2. Ordered groups

A *(totally) ordered group* is a partially ordered group $(G, \le)$ in which every pair of elements is comparable, that is, for all $x, y \in G$, either $x \le y$ or $y \le x$. Of course, the subgroups of $\mathbf{R}$ are ordered groups, but the simplicial group $\mathbf{Z}^n$ is not, unless $n = 1$.

If G and H are ordered groups, we can endow the product group $G \times H$ with a natural order making it an ordered group. Define $(x,y) \le (x',y')$ to mean that either $x < x'$, or $x = x'$ and $y \le y'$. This order is called the *lexicographic* order, and when endowed with it, $G \times H$ is called the *lexicographic* product. In a similar manner, one can define the lexicographic product of ordered groups $G_1, \ldots, G_n$, or indeed, of an infinite sequence $(G_n)_{n=1}^{\infty}$ of ordered groups.

The same group may admit different total orders. For instance, $\mathbf{Z}^2$ has the lexicographic order, and also, if θ is an irrational number, it has another order, namely the one whose cone is the set of all elements (m,n) such that $m+\theta n \geq 0$.

Note that an ordered group is necessarily torsion free. Also, it is not hard to show that a torsion-free partially ordered group G is an ordered group if and only if G^+ is a maximal cone. In fact, any torsion-free abelian group can be made into an ordered group, and we are therefore assured of a large supply of examples of ordered groups. (This result is due to Levi [7].)

Let G be a discrete abelian group, and denote by $\hat{G}$ its Pontryagin dual group. If x is a non-zero element of G of finite order, the set $\{\gamma(x) \mid \gamma \in \hat{G}\}$ is finite, and not a singleton, so it is disconnected. Hence, $\hat{G}$ is disconnected. Conversely, if $\hat{G}$ is disconnected, one can show that G admits a non-zero element of finite order ([13], p47). Thus, G is torsion free if and only if $\hat{G}$ is connected. This turns out to be important in the theory of Toeplitz operators defined relative to ordered groups, which we now discuss briefly.

If G is an ordered group, the *Hardy space* $H^2(G)$ is the L^2 norm closed linear subspace of $L^2(\hat{G})$ consisting of all functions f whose Fourier transform $\hat{f}$ is supported in G^+. If $G = \mathbf{Z}$, one gets the classical Hardy space on the circle group $\mathbf{T}$. Much of the original H^p space theory has been extended to the more general situation by Helson and Lowdenslager—for a detailed account see Rudin [13].

Let P denote the orthogonal projection of $L^2(\hat{G})$ onto $H^2(G)$. If φ is a complex-valued continuous function on $\hat{G}$, then the bounded linear operator

$$H^2(G) \to H^2(G), \quad f \mapsto P(\varphi f),$$

is denoted by T_φ, and called the *Toeplitz* operator with *symbol* φ. Using the fact that $\hat{G}$ is connected, the author gave a new proof of a result of Widom and Devinatz which asserts that the spectrum of T_φ is connected [10]. An important question concerning T_φ is its invertibility. In the classical case ($G = \mathbf{Z}$) it is well known that invertibility of T_φ is equivalent to the existence of a continuous logarithm for φ. The author extended this result to the general case [10], and here again the proof uses the connectivity of $\hat{G}$.

Denote by $\mathbf{A}(G)$ the C^*-algebra of operators on $H^2(G)$ generated by the Toeplitz operators. This algebra, and certain of its subalgebras, turn out to be interesting from the point of view of C^*-algebra theory. For instance, one gets a new class of simple C*-subalgebras (a C^*-algebra is *simple* if it has no non-trivial closed ideals—it is important to have examples of such algebras, but they are not always easy to obtain). The K-groups of these algebras are difficult to compute and have received much attention recently. For subgroups of the reals, the K-theory has been completely computed, and for general ordered groups some important partial information has been obtained. We shall return to this topic later, in connection with dimension groups.

An *archimedean* group is an ordered group G such that for every pair $x, y > 0$ there exists a positive integer n such that $x \leq ny$. The subgroups of $\mathbf{R}$ are clearly archimedean, and in fact these are all the archimedean groups, up to ordered group isomorphism (for a proof see [13]). If G is an archimedean group, then the commutator ideal of $\mathbf{A}(G)$ is simple, a result due to Douglas [3]. (The *commutator* ideal is the smallest closed ideal containing all of the additive commutators $ab - ba$.) The author showed that the converse is also true—if $\mathbf{A}(G)$ has simple commutator ideal, then G is archimedean. Thus, order properties of the group are strongly reflected in algebraic properties of its associated C^*-algebra, and conversely.

In analysing the C^*-algebras $\mathbf{A}(G)$, the author discovered the following result concerning ordered groups, which may be new: *A finitely-generated ordered group is a lexicographic product of a finite number of archimedean groups.* This does *not*, by any means, reduce the study of the algebras $\mathbf{A}(G)$ to the case where G is a subgroup of $\mathbf{R}$, but it is a useful result in the theory of these algebras (see [11]).

3. Dimension groups

The groups of the title of this section form a class of partially ordered groups which arise in the study of certain C^*-algebras, namely AF-algebras. They have been the subject of intensive study, and now have a fairly well-developed theory. For a comprehensive treatment, see Goodearl's recent AMS monograph [6]. Dimension groups are also covered in [1], [4] and [5].

An *AF-algebra* is a C^*-algebra A having an increasing sequence of finite-dimensional C^*-subalgebras A_n whose union $\cup_n A_n$ is dense in A. An example of such an algebra is the set of all compact operators on a separable Hilbert space. On the other hand, the C^*-algebra of all bounded operators is not an AF-algebra, unless the Hilbert space is finite-dimensional. The class of AF-algebras is sufficiently close to that of the finite-dimensional C^*-algebras to be tractable, but it is nevertheless a highly non-trivial class and exhibits typical C^*-algebra behaviour. Some C^*-algebras which are important in the theory of quantum mechanics belong to this class.

For the sake of simplicity, we shall only consider unital AF-algebras.

If A is a finite-dimensional C^*-algebra, it is easy to see that for some n, its K-group $K_0(A)$ is equal to a simplicial group $\mathbf{Z}^n$. If now A is assumed to be an AF-algebra, then by definition, it is a direct limit of finite-dimensional C^*-algebras, and therefore by continuity of the functor K_0, the partially ordered group $K_0(A)$ is the direct limit of a sequence of simplicial groups. These groups, direct limits of simplicial groups, are called *dimension groups*. (The positive cone of a K_0-group is thought of as the set of "dimensions" of the projections of the algebra, and of its matrix algebras.)

It is a remarkable, and very important, result of this theory that dimension groups have been given a very nice abstract and *usable* characterization:

Theorem 3.1 *A countable partially ordered group G is a dimension group if and only if the following conditions are satisfied:*

(1) If $nx \geq 0$ and $n > 0$, then $x \geq 0$;
(2) If $x_i \leq y_j$ for $i, j = 1, 2$, then there exists $z \in G$ such that $x_i \leq z \leq y_j$.

Condition (2) is called the *Riesz interpolation property*. The theorem is due to E. Effros, D. Handelman, and C.-L. Shen. For a proof see [5].

A dimension group closely reflects the structure of its corresponding AF-algebra. For instance, the closed ideals of the algebra correspond to certain subgroups of the dimension group, called its "ideals." Thus, if the dimension group is *simple*, that is, has no non-trivial ideals, the AF-algebra is simple.

If one wants to construct an AF-algebra with certain unusual properties, one may be able to do this by interpreting the properties in terms of the dimension group, and trying to construct the latter. In an important instance where this approach has been taken, and has paid off very well, B. Blackadar obtained a certain AF-algebra with unusual properties, from which he in turn constructed a C^*-algebra which is simple, yet has no non-trivial projections (self-adjoint idempotent elements). This solved a difficult problem which had been open for many years.

Since simple dimension groups are particularly important, we give some examples to illustrate the possibilities.

Every countable subgroup of $\mathbf{R}$ is a simple dimension group.

Let $G = \mathbf{Q}^n$, and define the positive cone to be

$$G^+ = \{(x_1, \ldots, x_n) \mid x_1, \ldots, x_n > 0\} \cup \{(0, \ldots, 0)\}.$$

The corresponding partial order on G is called the *strict* order. It is easy to see that G is a simple dimension group.

If $G = \mathbf{Q}^2$, where the positive cone is $G^+ = \{(x, y) \mid x > 0\} \cup \{(0, 0)\}$, then G is a simple dimension group.

If A and B are AF-algebras, under what conditions on their dimension groups are they isomorphic? The answer, due to G. Elliott, is easy to state, but the proof is difficult. A necessary and sufficient condition for A and B to be isomorphic is that there is an order isomorphism of the corresponding dimension groups, which is *unital* in the sense that the K_0-classes of the units of A and B correspond. If the dimension groups are only order isomorphic (with no assumption that the isomorphism is unital), then the algebras are *stably isomorphic*, which may be loosely asserted to mean they have the same representation theory.

We finish up by returning briefly to the theory of Toeplitz operators. Let G be an ordered group. If $\varphi = (\varphi_{ij})$ is a square matrix of order n whose entries are continuous complex-valued functions on $\hat{G}$, define the Toeplitz operator T_φ to be the matrix $(T_{\varphi_{ij}})$ (as an operator this acts on the orthogonal direct sum of n copies of $H^2(G)$). It is shown by the author in [12] that if T_φ is invertible, then φ is invertible and its class (its "topological index") in the K-group $K_1(C(\hat{G}))$ is the zero element. This extends a result known for $\mathbf{Z}$, and more generally, for subgroups of $\mathbf{R}$. The proof involves K-theoretic computations which use the fact that ordered groups are dimension groups, and therefore may be written as direct limits of simplicial groups. Actually, rather more is proved, and the interested reader is referred to [12] for details.

References

[1] B. Blackadar, K-Theory for Operator Algebras, (MSRI publications no. 5). Springer-Verlag: New York, 1986.

[2] A. Devinatz, *Toeplitz operators on H^2-spaces,* Trans. Amer. Math. Soc. **112** (1964), 304–317.

[3] R. G. Douglas, *On the C*-algebra of a one-parameter semigroup of isometries,* Acta Math. **128** (1972), 143–152.

[4] E. G. Effros, Dimensions and C*-algebras, (CBMS Regional Conf. Series in Math. no. 46). Amer. Math. Soc.: Providence, Rhode Island, 1981.

[5] K. R. Goodearl, Notes on Real and Complex C*-algebras. Shiva Publishing: Nantwich, 1982.

[6] K. R. Goodearl, Partially Ordered Abelian Groups with Interpolation, (AMS Mathematical Surveys and Monographs no. 20). Amer. Math. Soc.: Providence, Rhode Island, 1986.

[7] F. Levi, *Ordered groups,* Proc. Indian Acad. Sci. **16** (1942), 256–63.

[8] G. J. Murphy, *Ordered groups and Toeplitz algebras,* J. Operator Theory **18** (1987), 303–326.

[9] G. J. Murphy, C*-algebras and Operator Theory. Academic Press: New York, 1990.

[10] G. J. Murphy, *Spectral and index theory for Toeplitz operators,* Proc. Royal Irish Acad. **91A** (1991), 1–6.

[11] G. J. Murphy, *Toeplitz operators and algebras,* Math. Zeit. **208** (1991), 355–62.

[12] G. J. Murphy, *Almost-invertible Toeplitz operators and K-theory,* J. Integral Equations and Operator Theory **15** (1992), 72–81.

[13] W. Rudin, Fourier Analysis on Groups. Intersience: New York, London, 1962.

Gerard J. Murphy,
Department of Mathematics,
University College,
Cork.

HOW TO COMPOSE A PROBLEM FOR THE INTERNATIONAL MATHEMATICAL OLYMPIAD?

Fergus Gaines

A difficult task for the organizers of any national team for participation in the International Mathematical Olympiad (IMO) is to fulfill the request of the host nation to submit original problems for consideration by the jury for inclusion in the Olympiad. To compose such problems requires considerable skill and even mathematicians of a high calibre can find the task difficult because many of the techniques of the professional mathematician are excluded by the requirement that the problems have "elementary" solutions. Arthur Engel has written a very interesting article [2] describing his thought processes in composing IMO problems. Since the author is actively involved in the preparation and training of the Irish IMO team he has felt it incumbent on himself to compose suitable problems. This article describes some of his attempts.

In this article four avenues of approach to the task of composing IMO problems are considered. They are:
§1. Use a known result in some area of mathematics that might reasonably be assumed to be outside the knowledge of the contestants.
§2. Do a variation on a known elementary, but tricky, result.
§3. Compose a problem from a topic being currently taught by the composer.
§4. Use someone else's problem!

§1. In past Olympiads some of the problems which appeared were direct applications of a piece of mathematics which is well-known

to mathematicians, e.g. the pigeon-hole principle or the concept of an eigenvalue. But nowadays it is taken for granted that such problems would be deemed too trivial, because of the training that many of the contestants receive. The first idea the author had for constructing an Olympiad problem was to take a known result in some area of mathematics that might be reasonably assumed to be outside the knowledge of the contestants and to vary it a little. Thus, on page 3 of Jacobson's book [3] on Jordan algebras is the following result of Hua Loo Keng:

Theorem. *Let σ be an additive mapping of a division ring Δ into a division ring Δ' which preserves inverses. Then σ is either a homomorphism or an antihomomorphism.*

The question we ask is: does this give a non-trivial problem for the real numbers? As an answer we have

Problem 1. Let f be a function from the real numbers to the real numbers such that $f(1) = 1$, $f(a+b) = f(a) + f(b)$ for all a and b and $f(a)f(1/a) = 1$ for all $a \neq 0$. Prove that $f(x) = x$ for all x.

Proof. It is easy to prove that the properties $f(1) = 1$ and $f(a+b) = f(a) + f(b)$ for all a and b imply $f(x) = x$ for all rational numbers x. It is also not difficult to prove that f is injective and that $f(-x) = -f(x)$ for all x.
Next we note that, if $f(a) \neq f(a^2)$,

$$\begin{aligned} 1/\left[f(a) - f(a^2)\right] &= 1/\left[f\left(a(1-a)\right)\right] \\ &= f\left[1/a + 1/(1-a)\right] \\ &= 1/[f(a) - f(a)^2]. \end{aligned}$$

Thus $f(a^2) = f(a)^2$ here and this result is still true when $f(a) = f(a^2)$. Thus $f(x) > 0$ if $x > 0$. So $a > b$ implies $f(a) > f(b)$.

Finally, if x is any real number there exist two sequences $\{a_n\}$ and $\{b_n\}$ of rational numbers such that x is the only real number satisfying the condition $a_n < x < b_n$ for all natural numbers n. Then $a_n = f(a_n) < f(x) < f(b_n) = b_n$ holds for all n and hence $f(x) = x$ for all real numbers x.

This problem was submitted to the 30th IMO in Germany in 1989 but was not shortlisted. Perhaps it was not suitable because a characterization of the real numbers might not be known to some contestants.

Another, more recent, result that it was felt might yield a suitable problem is the following result of Leep and Shapiro [5]:

Theorem. *Let G be a subgroup of index 3 of the multiplicative group of a field F. Then every element of F is expressible in the form $g+h$ where g and h are elements of G, except when $|F| = 4$, 7, 13 or 16.*

Replacing F by the rational numbers doesn't seem to make the theorem any easier and, in any case, one shouldn't expect too many of the IMO contestants to know much about groups. But the theorem is the motivating idea for the following problem.

Problem 2. Let $\mathbf{Q}$ denote the set of rational numbers. Let S be a nonempty subset of $\mathbf{Q}$ with the properties:

(i) $0 \notin S$;

(ii) if $s_1, s_2 \in S$ then $s_1/s_2 \in S$;

(iii) there exists $q \in \mathbf{Q}$ with $q \neq 0$ such that every nonzero rational number not in S is of the form qs for some $s \in S$.

Prove that if $x \in S$ then there exist $y, z \in S$ such that $x = y + z$.

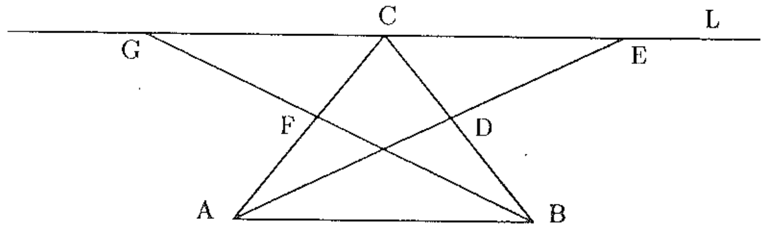
This problem is too easy for an IMO but it was included in the 1991 Irish Mathematical Olympiad and gave a lot of difficulty to the contestants because of the group theory concepts involved.

§2. Another idea for composing a problem is to take a known elementary, but tricky, result and do a variation of it. For example, the following is a well-known, difficult result in plane geometry.

The Steiner-Lehmus Theorem. *[1]. If the bisectors of two angles of a triangle are equal in length then the triangle is isosceles.*

The following variation suggested itself.

Problem 3. Let ABC be a triangle with L a line through C parallel to the side AB. Let the internal bisector of the angle at A meet BC at D and L at E, and let the internal bisector of the angle at B meet AC at F and L at G. If $DE = FG$ prove that $CA = CB$.



Proof. Let $BC = a$, $CA = b$, $AB = c$ and $A = 2\alpha$, $B = 2\beta$. Obviously

$$\frac{CF}{FA} = \frac{a}{c}, \qquad CF = \frac{ab}{a+c}, \qquad CD = \frac{bc}{b+c}.$$

Suppose $a > b$. Then $\alpha > \beta$, $\sin\alpha > \sin\beta$ and $\sin 2\alpha > \sin 2\beta$. In the triangle CFG

$$\frac{GF}{\sin 2\alpha} = \frac{CF}{\sin\beta}$$

and thus

$$GF = \frac{ab\sin 2\alpha}{(a+c)\sin\beta}.$$

In the triangle CDE

$$\frac{CD}{\sin\alpha} = \frac{ED}{\sin 2\beta}$$

and thus

$$ED = \frac{ab\sin 2\beta}{(b+c)\sin\alpha}.$$

Since $GF = ED$ we get

$$(b+c)\sin\alpha\sin 2\alpha = (a+c)\sin\beta\sin 2\beta.$$

Using the fact that $\dfrac{a}{\sin 2\alpha} = \dfrac{b}{\sin 2\beta}$ we get

$$\frac{\sin\alpha}{\sin\beta} = \frac{(a+c)b}{(b+c)a} = \frac{ab+bc}{ab+ac} < 1$$

since $a > b$. This implies $\sin\alpha < \sin\beta$, which contradicts the assumption $a > b$. It follows that $a \le b$. In the same way, $a \ge b$. Thus $a = b$.

Proofs of this result using Euclidean geometry and coordinate geometry have also been found.

This problem made it to the short list at the 31st IMO in Beijing in 1990 but did not feature in the final Jury discussions.

§3. Another area of inspiration for composing IMO problems is whatever the composer is teaching at the time! Thus, in teaching a course on complex analysis the author felt that the topic of Möbius transformations should yield a tricky problem. And, sure enough, we have

Problem 4. Let P be the set of positive rational numbers and let the function f from P to itself have the properties

(i) $f(x) + f(1/x) = 1$ and

(ii) $f(2x) = 2f(f(x))$ for all $x \in P$.

Determine, with proof, a formula for $f(x)$.

Proof. Let $x = 1$ in (i) to get $f(1) = \dfrac{1}{2}$.

Then (ii) yields $f(2) = 2f\left(\dfrac{1}{2}\right)$ and, putting $x = 2$, (i) gives $f(\frac{1}{2}) = \dfrac{1}{3}$ and $f(2) = \dfrac{2}{3}$. Trying a few more values of x leads one to suspect that $f(x) = \dfrac{x}{x+1}$ for all $x \in P$. If $x \in P$ then x can be written as $\dfrac{m}{n}$ where m and n are relatively prime natural numbers and we shall assume that all the rational numbers we deal with are expressed in this reduced form. Let $h(x) = m + n$ where $x = \dfrac{m}{n}$ is in reduced form. We prove that $f(x) = \dfrac{x}{x+1}$ by induction on $h(x)$.

It is clear that $\dfrac{x}{x+1}$ satisfies properties (i) and (ii) of the problem.

Next, $h(x) = 2$ forces $x = 1$ and $h(x) = 3$ forces $x = \dfrac{1}{2}$ or 2. Thus we have already verified the formula for $f(x)$ when $h(x) \le 3$. So let $x \in P$ with $h(x) > 3$ and assume the formula for $f(y)$ holds for all $y \in P$ with $h(y) < h(x)$.

Let $x = \dfrac{m}{n}$ be in reduced form and suppose m and n are both *odd*. Suppose also, without loss of generality, that $m < n$. Since

$$h\left(\frac{m}{n-m}\right) = n < m + n = h(x)$$

we have

$$f\left(\frac{m}{n-m}\right) = \frac{m}{n}.$$

There exists a natural number d such that $n - m = 2d$. Thus

$$\begin{aligned} f(x) &= f\left(f\left(\frac{m}{n-m}\right)\right) \\ &= \frac{1}{2}f\left(\frac{2m}{n-m}\right) \\ &= \frac{1}{2}f\left(\frac{m}{d}\right) \\ &= \frac{m}{2(m+d)} \end{aligned}$$

by the induction hypothesis, since

$$\begin{aligned} h\left(\frac{m}{d}\right) &= m + \frac{1}{2}(n-m) \\ &= \frac{m+n}{2} < h(x). \end{aligned}$$

Hence

$$f(x) = \frac{m}{m+n} = \frac{x}{x+1}.$$

Thus the formula holds if m and n are both odd.

Now suppose one of m and n, m say, is even. Then $m = 2^k r$ for some integer $k \geq 1$ and some odd integer r. Then we get

$$\begin{aligned} f(x) &= f\left(\frac{2^k r}{n}\right) \\ &= 2f\left(f\left(\frac{2^{k-1}r}{n}\right)\right) \\ &= 2f\left(\frac{2^{k-1}r}{2^{k-1}r+n}\right) \\ &= \\ &\vdots \quad \vdots \\ &= \\ & 2^k f\left(\frac{r}{(2^k-1)r+n}\right) \end{aligned}$$

and we note that

$$h\left(\frac{r}{(2^k-1)r+n}\right) = h(x).$$

Letting $m_1 = m$ and $n_1 = n$ we have proved that there exist natural numbers m_2 and n_2, with m_2 even and n_2 odd, so that $f(m_1/n_1) = 2^k f(n_2/m_2)$ and $h(m_1/n_1) = h(m_2/n_2)$. We then have $f(m_1/n_1) = 2^k[1 - f(m_2/n_2)]$. Repeating this process we get a sequence of positive rationals m_i/n_i with m_i even and n_i odd, $f(m_i/n_i) = 2^{k_i}[1 - f(m_{i+1}/n_{i+1})]$, where k_i is the highest power of 2 dividing m_i and $h(m_i/n_i) = h(x)$ for $i = 1, 2, \ldots$

Since there are only finitely many rationals satisfying the last condition there exist natural numbers r and s with $r < s$ so that $m_r/n_r = m_s/n_s$. Then there exist integers p and q so that $f(m_r/n_r) = p + qf(m_s/n_s)$ and $q = \pm 2^t$ for some natural number t. Hence we get the unique value $p/(1-q)$ for $f(m_r/n_r)$. Thus we get a unique value for $f(x)$. Since only properties (i) and (ii) were used in deriving this value of $f(x)$, and $\frac{x}{x+1}$ satisfies these properties, uniqueness implies that $f(x) = \frac{x}{x+1}$. So the result is true by induction.

It is probably clear how this problem was composed: write down a few properties of the function $\frac{x}{x+1}$ and try to recover the original function from these properties.

This problem was also included in the 1991 Irish Mathematical Olympiad. It was intended originally to submit the problem for consideration by the IMO jury in Sweden in 1991 but, owing to an error in the author's original solution, the problem was considered too easy. T. J. Laffey supplied the crucial argument for dealing with $\frac{m}{n}$ when m is even and n is odd.

It can happen with IMO-type problems that a very elegant solution can be produced which the composer did not envisage (The author would welcome such a solution to Problem 4 above!). At the 30th IMO in Braunschweig the following problem was accompanied by a very complicated solution and was rated A++ by the jury:

Problem 5. A permutation $(x_1, x_2, \ldots, x_{2n})$ of the set $\{1, 2, \ldots, 2n\}$, where n is a positive integer, is said to have property P if $|x_i - x_{i+1}| = n$ for at least one i in $\{1, 2, \ldots, 2n-1\}$. Show that, for each n, there are more permutations with property P than without.

My colleague Mícheál Ó Searcóid came up with this elegant proof of a more general result:

We say that the permutation $(x_1, x_2, \ldots, x_{2n})$ has an an *adjacent pair* if and only if $|x_i - x_{i+1}| = n$ for some i with $1 \leq i \leq 2n-1$. Let S be the set of those permutations with *exactly one* adjacent pair and let T be the set of permutations with no adjacent pairs. We shall prove that $|S| > |T|$. This is clear if $n = 1$. So let $n > 1$. Let $f : T \to S$ such that $f(x_1, \ldots x_{2n}) = (x_2, x_3, \ldots, x_{j-1}, x_1, x_j, \ldots, x_{2n})$ where $|x_1 - x_j| = n$. Then f is well-defined and injective. It is not surjective since, for example, the permutation $(1, n+1, 2, 3, \ldots n, n+2, \ldots 2n)$ is in S but not in T. Hence $|S| > |T|$. This more general result now implies the proof of problem 5.

§4. One final way of getting a problem for the IMO is to use someone else's! The other "Irish" problem shortlisted for the IMO in Beijing is the creation of Charles Johnson of the College of William and Mary in Virginia. It is:

Problem 6. An eccentric mathematician has a ladder with n rungs which he ascends and descends in the following way: whenever he ascends each step he takes covers a rungs of the ladder and whenever he descends each step he takes covers b rungs of the ladder. By a sequence of ascending and descending steps he can climb from ground level to the top rung of the ladder and climb down to ground level again. Find, with proof, the smallest value of n, expressed in terms of a and b.

Solution. The smallest value of n is $a+b-(a,b)$ where (a,b) is the greatest common divisor of a and b. This is obvious if $a|b$ or $b|a$.

Suppose that $(a,b) = 1$. Suppose also that $a > b$ (there is no loss of generality since the problem is symmetric with respect to ascending or descending the ladder). Then there exist natural numbers r_1, s_1 so that

$$a = bs_1 + r_1$$

where $0 < r_1 < b$. In general, given the remainder r_{j-1}, there exist integers r_j and s_j so that $a + r_{j-1} = bs_j + r_j$, where $0 \le r_j \le b-1$, for $j = 2, 3, \ldots$. Since $a \equiv r_1 \pmod b$ we get $r_j \equiv jr_1 \pmod b$, for $j = 1, 2, \ldots$. Since $(r_1, b) = 1$ the integers $r_1, r_2, \ldots, r_b$ are distinct and thus are equal to $0, 1, 2, \ldots, b-1$ in some order. We must have $r_b = 0$, since $r_j = 0$, for some $j < b$, implies that $r_{j+1} = r_1$, which is a contradiction. If the mathematician is standing on rung r_j, (counted from the bottom) of the ladder and $a + r_j \le n$ then, by ascending by a rungs and descending by bs_{j+1} rungs, he can get to rung r_{j+1}. So, if $a+b-1 = n$, we have $a + r_j \le n$ for $j = 1, 2, \ldots, b$ and, since $b - 1 = r_j$ for some j, he can clearly get to rung r_j for each $j = 1, 2, \ldots, b$ and thus he can climb to the top rung of the ladder and back to ground level again. If $n < a+b-1$ he can not reach rung r_j for some $j \le b$ and thus he can not reach "rung" r_b, i.e. ground level, and thus he can not ascend and descend the ladder in the required way. So the smallest value of n is $a+b-1$. Finally, if $(a,b) = k > 1$ replace a and b in the above discussion by a/k and b/k, respectively, and then scale all integers up by k to get $a+b-(a,b)$ as the smallest value of n.

References

[1] H. S. M. Coxeter and S. L. Greitzer, Geometry Revisited. The Mathematical Association of America: Washington D.C., 1967.

[2] Arthur Engel, *The creation of Mathematical Olympiad problems*, Newsletter of the World Federation of National Mathematics Competitions **5** (1987).

[3] N. Jacobson, Structure And Representations Of Jordan Algebras. American Mathematical Society: Providence, Rhode Island, 1968.

[4] M. S. Klamkin (compiler), International Mathematical Olympiads 1979-1985. The Mathematical Association of America: Washington D.C., 1986.

[5] D. B. Leep and D. B. Shapiro, *Multiplicative subgroups of index three in a field,* Proc. Amer. Math. Soc. **105** (1989), 802–807.

Fergus J. Gaines,
Department of Mathematics,
University College Dublin,
Dublin.

THE USE OF THE COMPUTER IN MATHEMATICS TEACHING PAST HISTORY — FUTURE PROSPECTS

D. J. Evans

Initially, the digital computer was created to facilitate the solution of problems that required numerical computations which were, for all practical purposes, beyond or impossible under previous technologies. Subsequent developments of MAINFRAME COMPUTING have been similarly motivated by the need to solve increasingly complex scientific problems or by the need to manipulate huge quantities of information.

Digital computers first made their appearance on college and university campuses in the late 1950s. At that time virtually all computing was done in batch mode and programming using a scientific programming language such as FORTRAN or possibly a local dialect. Under these conditions, the time from submission to job return would be not less than 45 minutes. The forbidding nature of FORTRAN syntax and the complexity of its input-output statements required that a more "user friendly" language be created. As a result, alternative languages such as BASIC and WATFOR were created and extensively used by students.

The next major development in computing which affected teaching use was INTERACTIVE COMPUTING. Under this mode of operation, a user could be connected to a mainframe from a remote location, enter a program, and execute it. As a result the "turnaround" time between jobs became less than 5 minutes.

New applications of the computer to instruction and teaching also became possible soon after. Perhaps the major new development of concern to us today was COMPUTER AIDED INSTRUCTION (CAI) which quickly became a focal point of concern for

education researchers. The output from most of the programs at that time was still numerical or perhaps consisted of "computer graphics". Some instructors, however, began to use GRAPHICS terminals and their impact was immediately felt.

With the introduction of the MICROCOMPUTER in the late 1970s, graphics became widely available. More importantly, the microcomputer changed the locus of control from the campus computer center running large mainframes in a multiuser environment to a departmental computer laboratory controlled by the local teaching staff. With the introduction of the IBM personal computer and the Apple Macintosh processor, bit mapped screens soon became available for sophisticated graphics use as the amount of random access memory increased from 48K to 512K or more and the processors operated very much faster. The declining price for computers also made it possible for most high schools to have computers available for their students.

The developments I have outlined are of ever changing technology driven primarily by non-instructional needs. Nevertheless, as technology changes the instructional applications will also change. As we look to the future, the one thing we can be certain of is that the technology will continue to advance, and that computers will continue to decline in price. Thus we have today an environment in which our students arrive at college knowing how to use a computer and able to purchase a microcomputer for approximately £200 with a capability greater than many of the early mainframes. In addition, hand held calculators are now readily available with a programming and graphics capability equivalent to the early microcomputers. There can be no doubt that this technology will affect both the way we teach UNDERGRADUATE MATHEMATICS and the COURSE CONTENT.

As we look to the future, we see an ever changing technology which will continue to provide opportunities for teaching innovation. The advanced workstations of today will become the commonplace equipment of tomorrow. These machines will be linked by networks which will access fileservers, high quality printers, and gateways to external resources such as remote data bases and libraries. So instead of focusing on the use of computers to improve the teaching of current mathematical topics, we can begin to explore simulated learning environments which will enable us to teach subject material previously thought to be too abstract or complex for undergraduates.

Computer Impact on Mathematics

In Mathematics computers first made their impact on *Numerical Analysis* and during the period 1956–75 the subject developed greatly, both in stature and discipline. *Statistics,* also being a numerate topic, quickly followed the same fate. It has only been recently, since the introduction of the more powerful supercomputer and parallel computers that we have seen a similar impact on *Algebra,* to be followed shortly with improved graphics and visualization technology on *Geometry.*

Discrete Mathematics

Discrete mathematics is what computers actually do. Therefore it should be compulsory to all our students if they are to achieve some proficiency and affinity with computers. It is the mathematics of finite and countable sets, and it includes topics taught throughout the standard secondary and college curricula. These topics include logic, set theory, combinatorics, discrete probability, functions and relations on discrete structures, induction, recursion, difference equations, graph theory, trees, algebraic structures, and linear algebra.

Finite Differences

The computer age has given new impetus to the method of finite differences, which treats problems of time evolution posed in discrete rather than continuous form. This is an old subject studied by Boole in the 19th century.

A discrete mathematics centered on difference equations is timely, not only because of the increased usage of computers, but because of the known shortcomings in the Infinitesimal Calculus.

The Calculus of Newton and Leibniz was designed to circumvent the difficulties of dealing with the discrete by the passage

to continuous limits. Sums became integrals, differences became differentials, and the computational labour of repeated additions and subtractions avoided by the power of the Infinitesimal Calculus. Another difficulty with traditional calculus is that many modern problems resist continuous methods. Even the student who succeeds with calculus needs to appreciate discrete approximation schemes which can be implemented on the computer. Such schemes are increasingly important for the solution of the analytically intractable differential equations arising in many applications in the real world.

Discrete mathematics also creates a link between ideas and techniques of computer science and important and useful mathematical notions. An elementary course on the subject can even bring students into contact with active research in mathematics, physics, chemistry, and other areas.

Linear Algebra

The topic which has been most affected by present day computers is Linear Algebra. What then may the main topics in a Linear Algebra course of the future become? Surely the elementary theory concerning the concepts of linear independence, span, basis, and dimension will remain fundamental, and properties of the algebra of matrices and linear transformations will not lose their importance. Also the geometry of vectors will continue to provide important insights and examples. The analysis of linear systems of equations and the investigation of eigenvectors and eigenvalues will require added emphasis, since the computer software allows us to ask so many more interesting questions involving these objects.

However, reduction methods and echelon forms will need a different approach, since the software and algorithms which carry out the necessary computation are often quite different from those now taught. Without doubt, reduction to upper triangular form, possibly using partial pivoting (and followed by back substitution if there is an equation to be solved) should be the main hand computation approach, as this is similar to the LU-decomposition that a good Linear Algebra software package uses.

Methods for computing matrix inverses and detailed discussions of determinants probably require less attention. Since applications rarely require the computation of a matrix inverse it is always better to solve the related linear system. Also determinants have decreased in importance since modern algorithms for approximating eigenvalues and eigenvectors make no use of them. Furthermore, two other applications of the determinant, Cramer's Rule and the adjoint formula for the matrix inverse have become even more superfluous since the computation is done by software which makes no use of them.

There should also be a subtle alteration in emphasis throughout the course. Instead of paying close attention to the elements of a matrix, a point of view that is reinforced by hand computation, the properties of the matrix as an entity should be stressed.

Finally, the major addition to the Linear Algebra course should be the study of applications. Interesting applications that lead to linear systems of equations should be studied, i.e. temperature distributions found by approximating values at discrete grid points, input-output models in economics, electrical circuit analysis, least-squares approximation, balancing chemical reactions, and network analysis. Applications that involve locating eigenvalues and eigenvectors include: Markov chains, biological population models, and models of genetic inheritance. If there is time to study first-order linear systems of differential equations, many more applications become within reach.

Numerical Solution of Systems of Equations

Methods of solving systems of *equations* are divided into (i) direct and (ii) indirect, or iterative, methods. For linear equations the direct methods include the well known Gaussian elimination process, the indirect methods include the Gauss-Seidel method.

The direct methods have the advantages (a) that they will always produce the solution provided that it exists, is unique and that sufficient accuracy is retained at each and every stage, and (b) that the solution is found after a known number of operations. They have the disadvantage that very sparse systems of equations, such as those which arise in finite difference/element approximations to differential equations, may become rapidly less

sparse as the elimination process proceeds so raising the storage requirement from a multiple of n (for n equations) to something like n^2.

The iterative methods, on the other hand, may fail to converge to a solution and if they do converge it is not obvious how many operations they will require to produce the desired accuracy. They have however, the very considerable advantage that they are very well suited to computers and preserve the sparsity of the coefficient matrix throughout.

Direct methods for the numerical solution of *non-linear systems* are rarely available; there is, after all, no direct method for solving the general polynomial of even the fifth degree and so iterative methods are generally used. As in the case of linear systems, convergence may not always occur, though conditions sufficient to ensure convergence are usually known; and although in some cases the number of iteration necessary to produce convergence to a specified accuracy may not be easily predicted, it is frequently not a matter of great importance. However accelerating techniques can often be used if time is limited.

The revival of interest in iterative methods brought about by the use of computers has led to significant advances in the study of functions which are iteratively defined, e.g. by a nonlinear relation of the type

$$Z_{n+1} = F(Z_n)$$

where Z_0 is a given complex number and the function $F(Z)$ may contain one or more parameters. Some simple functions of this type are the quadratic equation

$$az^2 + bz + c = 0 \qquad (1)$$

By rearranging terms and a change of variable we can express this in quadratic iteration form, i.e.

$$Z_{n+1} = Z_n^2 + C, \qquad (2)$$

from which there are 3 possibilities:

1. The sequence Z_n converges to a limit α which is the solution of (1).
2. The sequence Z_n does not converge but the points Z_n remain bounded.
3. The points Z_n eventually move outside any bounded region.

In general all 3 cases can occur. Moreover, the complex values of C for which the sequence starting with $Z_0 = 0$ is either of type 1 or of type 2 form the well known Mandelbrot set, which has been the topic of much research recently.

Algorithms

An algorithm is simply a procedure for solving a specific problem or class of problems. The idea of an algorithm has been around for over 2000 years (e.g. the Euclidean Algorithm for finding the highest common factor of two integers) but it has attracted much greater interest in recent years following the introduction of computers and their application not only in mathematics but also to problems arising in technology, automation, business, commerce, economics, social sciences, etc.

Computer algorithms have been developed for many commonly occurring types of problem. In some cases several algorithms have been produced to solve the same problem, e.g. to sort a file of names into alphabetical order or to invert a matrix, and in such cases people who wish to use an algorithm will not only want to be sure that the algorithm will do what it is supposed to do, but also which of the several algorithms available is, in some sense, the "best" for their purposes. An algorithm which economizes on processor time may be extravagant in its use of storage space or vice-versa and the need to find algorithms which are optimal, or at least efficient, with respect to one or more parameters has led to the development of Complexity Theory. For instance the Fast Fourier Transform has reduced the time complexity from order n^2 to order $n \log n$, which is of considerable practical importance for large values of n.

An important aim in algorithm design is to ensure that the algorithm is "robust" i.e is guaranteed to produce the required answer under as wide a variety of conditions as possible.

However, the interest is still (and always has been) on reliable methods which always converge. For instance the quadratic equation (1) when a, b and c are real can be solved in many ways. However from the Theory of Equations we have the roots α_1, α_2 are given by

$$\alpha_1 + \alpha_2 = -\frac{b}{a} \quad \text{and} \quad \alpha_1\alpha_2 = \frac{c}{a}$$

which when written in iterative form

$$\alpha_1^{(n+1)} = \frac{b}{a} - \alpha_2^{(n)} \quad \text{and} \quad \alpha_2^{(n+1)} = \frac{c}{a\alpha_1^{(n+1)}} \tag{3}$$

where $\alpha_2^{(0)}$ is given, give a convergent algorithm and is preferable to the quadratic iteration form (2).

The direct methods too should be reconsidered in the demanding circumstances of present day software requirements. A simple program just using the formula

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

will just not do. A robust software package should check its input for validity as well as all the possible variations of the formula, i.e. when the roots are complex, etc. Also it must generate sufficient error messages so as to leave no doubt in the user's mind. Thus a complete flow diagram for the direct solution of the quadratic equation (1) is shown in Fig. 1.

Another area of great importance is the acceleration of the convergence of iterative processes especially for the large systems of linear equations which occur in scientific problems.

Given the trivial (2×2) system $Ax = b$ where A is symmetric and positive definite, i.e.

$$A = \begin{pmatrix} 1 & -a \\ -a & 1 \end{pmatrix}$$

QUADRATIC EQUATION $ax^2 + bx + c = 0$ DIRECT METHOD

Unseen input $c, b, a \rightarrow$

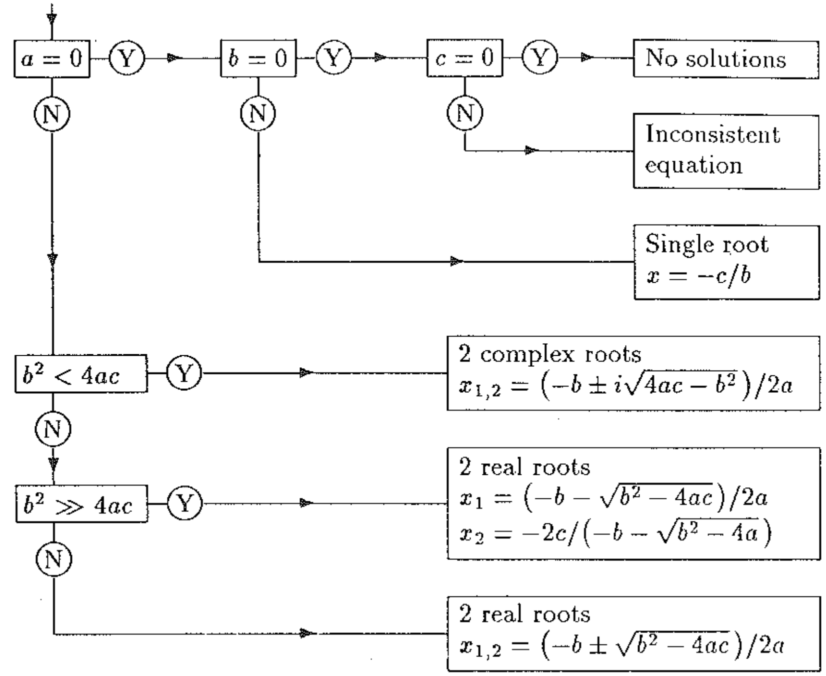


FIGURE 1

then the well known Gauss-Seidel method

$$x_1^{(k+1)} = a x_2^{(k)} + b_1$$
$$x_2^{(k+1)} = a x_1^{(k+1)} + b_2$$

is known to give acceptable convergence unless $a = 1$. It is usual to apply the successive overrelaxation (SOR) method, i.e.

$$x_1^{(k+1)} = x_1^{(k)} + \omega_1 \left(a x_2^{(k)} + b_1 - x_1^{(k)} \right)$$
$$x_2^{(k+1)} = x_2^{(k)} + \omega_1 \left(a x_1^{(k+1)} + b_2 - x_2^{(k)} \right)$$

where the overrelaxation parameter ω_1 is chosen from the formula

$$\omega_1 = \frac{2}{1 + \sqrt{1 - a^2}}$$

to obtain an extremely rapid convergence when a is very close to 1.

An even more rapid convergence is obtained when A is skew symmetric, i.e.

$$A = \begin{pmatrix} 1 & a \\ -a & 1 \end{pmatrix}.$$

Now the successive underrelaxation (SUR) method becomes

$$x_1^{(k+1)} = x_1^{(k)} + \omega_2 \left(-a x_2^{(k)} + b_1 - x_1^{(k)} \right)$$
$$x_2^{(k+1)} = x_2^{(k)} + \omega_2 \left(a x_1^{(k+1)} + b_2 - x_2^{(k)} \right)$$

where the optimal acceleration parameter ω_2 is now given by

$$\omega_2 = \frac{2}{1 + \sqrt{1 + a^2}}.$$

A comparison of the results for the 3 methods is given in Table 1, where N is the number of iterations required to achieve an accuracy of 10^{-6}.

a	Gauss-Seidel		SOR		SUR	
	$\omega = 1$	N	ω_1	N	ω_2	N
0.8090	1	33	1.2596	11	0.8748	7
0.9969	1	2,238	1.8545	88	0.8292	8
0.9988	1	5,732	1.9065	141	0.8288	8

TABLE 1: A comparison of the convergence rates of the Gauss-Seidel, SOR and SUR methods.

Recursive Algorithms

Algorithm design is an important topic which we should teach to our students — especially how to construct a recursive algorithm and in the following we show the details of a recursive parallel algorithm design.

In the numerical solution of partial differential equations by the implicit methods there occurs the problem of repeatedly solving linear systems involving tridiagonal matrices possessing diagonal dominance. Current algorithmic solution methods involve a Gaussian elimination of the matrix equation to upper triangular form with unit diagonal entries, from which the solution vector can be easily obtained by a back substitution process. In algorithmic form, we calculate the quantities

$$\begin{aligned} g_1 &= \frac{c_1}{b_1}, \quad g_i = \frac{c_i}{b_i - a_i g_{i-1}}, \quad i = 2, 3, \ldots, n-1 \\ h_1 &= \frac{d_1}{b_1}, \quad h_i = \frac{d_i - a_i h_{i-1}}{b_i - a_i g_{i-1}}, \quad i = 2, 3, \ldots, n, \end{aligned} \tag{4a}$$

and the solution is given by

$$x_n = h_n, \quad x_i = h_i - g_i x_{i+1}, \quad i = n-1, n-2, \ldots, 2, 1. \tag{4b}$$

However, it is well known that such back substitution processes (4b) are more ideally suited for serial computers and nowadays with the ever increasing usage of parallelism in algorithms it is necessary to investigate whether a more efficient parallel algorithm based on the Gauss-Jordan method can be formulated.

We now consider the $(n \times n)$ tridiagonal system given by

$$\begin{pmatrix} b_1 & c_1 & & & 0 \\ a_2 & b_2 & c_2 & & \\ & \ddots & \ddots & \ddots & \\ 0 & & \ddots & \ddots & c_{n-1} \\ & & & a_n & b_n \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ \vdots \\ d_n \end{pmatrix} \tag{5}$$

and since $b_i > a_i + c_i$, $i = 1, 2, \ldots, n$ then we are assured that no pivoting is required in any ensuing elimination process and hence

the tridiagonal matrix structure will be maintained in successive eliminations.

Initially the first equation is normalized by setting

$$g_1 = \frac{c_1}{b_1}, \quad h_{1,0} = \frac{d_1}{b_1}. \tag{6}$$

Then, the coefficient of x_1 in the second equation is eliminated by multiplying the first equation by $-a_2$ and adding to the second equation, i.e.

$$\begin{pmatrix} 1 & g_1 & & & \\ 0 & b_2 - a_2 g_1 & c_2 & & 0 \\ & a_3 & b_3 & c_3 & \\ 0 & & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \end{pmatrix} = \begin{pmatrix} h_{1,0} \\ d_2 - a_2 h_{1,0} \\ \vdots \\ \vdots \end{pmatrix} \tag{7}$$

which is then normalized by setting

$$g_2 = \frac{c_2}{b_2 - a_2 g_1}, \quad h_{2,0} = \frac{d_2 - a_2 h_{1,0}}{b_2 - a_2 g_1}.$$

From now on the Gauss-Jordan elimination proceeds differently and eliminates coefficients both below and above the diagonal as follows:

The coefficient of x_2 in the third equation is eliminated by multiplying the second equation by $-a_3$ and adding to the third equation, and the coefficient of x_2 in the first equation is eliminated by multiplying the second equation by $-g_1$ and adding to the first equation, i.e.

$$\begin{pmatrix} 1 & 0 & -g_1 g_2 & & & \\ 0 & 1 & g_2 & & & 0 \\ & & b_3 - a_3 g_2 & c_3 & & \\ & & a_4 & b_4 & c_4 & \\ 0 & & & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ \vdots \end{pmatrix} = \begin{pmatrix} h_{1,0} - g_1 h_{2,0} \\ h_{2,0} \\ d_3 - a_3 h_{2,0} \\ \vdots \\ \vdots \end{pmatrix} \tag{8}$$

which is similarly normalized by setting

$$g_3 = \frac{c_3}{b_3 - a_3 g_2}, \quad h_{3,0} = \frac{d_3 - a_3 h_{2,0}}{b_3 - a_3 g_2}, \quad \text{and} \quad h_{1,1} = h_{1,0} - g_1 h_{2,0}.$$

A further step of elimination illustrates further the pattern of the algorithm. The coefficient of x_3 in the fourth equation is eliminated by multiplying the third equation by $-a_4$ and adding to the fourth equation, the coefficient of x_3 in the second equation is eliminated by multiplying the third equation by $-g_2$ and adding to the second equation, the coefficient of x_3 in the first equation is eliminated by multiplying the third equation by g_1g_2 and adding to the first equation, i.e.

$$\begin{pmatrix} 1 & 0 & 0 & g_1g_2g_3 & & & \\ & 1 & 0 & -g_2g_3 & & 0 & \\ & & 1 & g_3 & & & \\ & & & b_4-a_4g_3 & c_4 & & \\ & 0 & & a_5 & b_5 & c_5 & \\ & & & & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ \vdots \\ \vdots \end{pmatrix} = \begin{pmatrix} h_{1,1}+g_1g_2h_{3,0} \\ h_{2,0}-g_2h_{3,0} \\ h_{3,0} \\ d_4-a_4h_{3,0} \\ \vdots \\ \vdots \end{pmatrix} \tag{9}$$

which is similarly normalized by setting

$$g_4 = \frac{c_4}{b_4 - a_4g_3}, \quad h_{4,0} = \frac{d_4 - a_4h_{3,0}}{b_4 - a_4g_3},$$
$$h_{1,2} = h_{1,1} + g_1g_2h_{2,0}, \quad \text{and } h_{2,1} = h_{2,0} - g_2h_{3,0} \tag{10}$$

By continuing in a similar manner for the rows $3, 4, \ldots, n$, it can be verified that ultimately the system (5) is transformed to the form

$$\begin{pmatrix} 1 & & & \\ & 1 & 0 & \\ & & \ddots & \\ 0 & & \ddots & \\ & & & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} h_{1,n-1} \\ h_{2,n-2} \\ \vdots \\ h_{n-1,1} \\ h_{n,0} \end{pmatrix} \tag{11}$$

from which we can see that the original tridiagonal system is now completely decoupled and the solution is immediately available.

Thus, to summarize the algorithmic process, we calculate the quantities

$$g_1 = \frac{c_1}{b_1}, \quad h_{1,0} = \frac{d_1}{b_1},$$

$$g_i = \frac{c_i}{b_i - a_ig_{i-1}}, \quad i = 2, 3, \ldots, n-1 \tag{12}$$
$$h_{i,0} = \frac{d_i - a_ih_{i-1,0}}{b_i - a_ig_{i-1}}, \quad i = 2, 3, \ldots, n.$$

Then, for $i = 3, 4, \ldots, n-1$, $k = i-2, i-3, \ldots, 2, 1$,

$$h_{k,i-k-1} = h_{k,i-k-2} + (-1)^{i-k-1} \prod_{j=k}^{j=i-2} g_jh_{i-1,0}.$$

Finally, the solution vector x is given by

$$x_i = h_{i,n-i}, \quad i = 1, 2, \ldots, n.$$

Thus, by using column sweep techniques which can be completed in parallel as the algorithm proceeds we are able to eliminate the recursive back substitution process completely from the computation.

The application of this direct method to the numerical solution of matrix equations arising from finite difference approximations to elliptic partial differential equations in two and higher dimensions can be made in the following manner. For two dimensional problems, these finite difference approximations produce matrix equations of the form $AX = D$, where the matrix A has the form

$$A = \begin{pmatrix} B_1 & C_1 & & & \\ A_2 & B_2 & C_2 & & 0 \\ & \ddots & \ddots & \ddots & \\ 0 & & \ddots & \ddots & C_{N-1} \\ & & & A_N & B_N \end{pmatrix}. \tag{13}$$

Here, the square submatrices B_j are of order n_j, where n_j corresponds to the number of mesh points on the jth horizontal mesh line of the discrete problem. The direct inversion method (12) can be immediately generalized so as to apply to $AX = D$. Indeed, if the vectors X and D are partitioned relative to the matrix A of

(13), then we define

$$G_1 = B_1^{-1}C_1, \quad H_{1,0} = B_1^{-1}D_1,$$
$$G_i = (B_i - A_iG_{i-1})^{-1}C_i, \quad i = 2,3,\dots,N-1 \tag{14}$$
$$H_{i,0} = (B_i - A_iG_{i-1})^{-1}(D_i - A_iH_{i-1,0}), \quad i = 2,3,\dots,N.$$

Then, for $i = 3,4,\dots,n-1$, $k = i-2, i-3, \dots, 1$,

$$H_{k,i-k-1} = H_{k,i-k-2} + (-1)^{i-k-1}\prod_{j=k}^{j=i-2} G_jH_{i-1,0},$$

and the vector components X_i of the solution are given by

$$X_i = H_{i,n-i}, \quad i = 1,2,3,\dots,n.$$

Parallel Algorithms

More recently the problem of designing algorithms which can be efficiently run on several processors working in parallel has attracted considerable interest. Algorithms which are ideal on a single processor may be highly inefficient, or even fail entirely on parallel processors and the design of suitable parallel algorithms for even the commonest problems is a matter for present day research.

Conclusions

The power of computers has given us the following opportunities:

i) to make new discoveries in Mathematics;
ii) in the teaching of Mathematics itself;
iii) to develop new methods (algorithms) which are efficient on computers for the solution of a wide range of problems and particularly so on parallel computers.

D. J. Evans,
Parallel Algorithms Research Centre,
University of Technology,
Loughborough,
Leicestershire LE11 3TU,
England.

CAUCHY'S MATRIX, THE VANDERMONDE MATRIX AND POLYNOMIAL INTERPRETATION

R. Gow

Let K be a field and let $\alpha_1, \dots, \alpha_n$ be elements of K. The $n \times n$ matrix $V = V(\alpha_1, \dots, \alpha_n)$, where

$$V = \begin{pmatrix} 1 & 1 & \dots & 1 \\ \alpha_1 & \alpha_2 & \dots & \alpha_n \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_1^{n-1} & \alpha_2^{n-1} & \dots & \alpha_n^{n-1} \end{pmatrix},$$

is called a Vandermonde matrix. It is an example of a type of matrix known as an *alternant*. See, for example, Chapter XI of [5]. The Vandermonde matrix plays an important role in problems concerning polynomials, symmetric polynomials in particular. The determinant of V is well known to be the difference product

$$\prod_{i>j}(\alpha_i - \alpha_j)$$

and thus V is invertible precisely when the α_i are all different. A proof that $\det V$ has the form stated above may be given as follows. Row operations show that $\det V$ equals the determinant of the $n \times n$ matrix obtained from V by replacing its last row by the row

$$(\, f(\alpha_1) \quad f(\alpha_2) \quad \dots \quad f(\alpha_n)\,),$$

where f is any monic polynomial in $K[x]$ of degree $n-1$. We choose f to equal

$$(x-\alpha_1)\dots(x-\alpha_{n-1}).$$

Then we have $f(\alpha_i) = 0$ for $i \neq n$ and

$$f(\alpha_n) = (\alpha_n - \alpha_1)\dots(\alpha_n - \alpha_{n-1}).$$

If W is the $n \times n$ matrix obtained from V for this choice of f, we easily see that

$$\det V = \det W = f(\alpha_n) \det V(\alpha_1, \dots, \alpha_{n-1})$$

and the result follows easily by induction. Occasionally, evaluations of $\det V$ in the literature seem to be unnecessarily complicated, as they refer to facts about homogeneous polynomials. The original evaluation of the determinant is due to Cauchy (Journal de L'École Polytechnique, XVII, 1815).

Let $\mathcal{P}$ denote the n–dimensional vector subspace of $K[x]$ consisting of all polynomials of degree at most $n-1$. The polynomials $1, x, \dots, x^{n-1}$ form the standard basis of $\mathcal{P}$. Let $p_1, \dots, p_n$ be n polynomials in $\mathcal{P}$. Then we may write

$$p_i = \sum_{k=1}^{n} a_{ik} x^{k-1},$$

where the a_{ik} are elements of K. If we evaluate the p_i at the points $\alpha_1, \dots, \alpha_n$, we obtain the matrix relation

$$P = AV,$$

where P is the $n \times n$ matrix whose (i,j) entry is $p_i(\alpha_j)$. Suppose that the α_i are all different, so that V is invertible. We choose the p_i to be the Lagrange interpolation polynomials for the points $\alpha_1, \dots, \alpha_n$, which are defined by the formulae

$$p_i = \frac{p}{p'(\alpha_i)(x - \alpha_i)} \text{ where } p = (x - \alpha_1)\dots(x - \alpha_n)$$

for $1 \leq i \leq n$. Then we find that $p_i(\alpha_i) = 1$ and $p_i(\alpha_j) = 0$ if $i \neq j$. Thus the matrix relation above becomes

$$I_n = AV$$

and the matrix A is the inverse of the Vandermonde matrix. We see that the coefficients of the interpolation polynomials enable us to find the inverse of V.

Suppose now that we have n additional different elements $\beta_1, \dots, \beta_n$ with $\alpha_i \neq \beta_j$ for all i and j. The $n \times n$ matrix

$$\begin{pmatrix} \frac{1}{\alpha_1-\beta_1} & \frac{1}{\alpha_1-\beta_2} & \cdots & \frac{1}{\alpha_1-\beta_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{\alpha_n-\beta_1} & \frac{1}{\alpha_n-\beta_2} & \cdots & \frac{1}{\alpha_n-\beta_n} \end{pmatrix}$$

is called a Cauchy matrix. The Cauchy matrix is an example of a *bialternant* or *double alternant*, as discussed in Chapter XI of [5]. It was introduced by Cauchy in a work [1, pp 151–159] published in 1841, where its determinant is calculated. The Cauchy matrix also appears briefly in Frobenius's development of the irreducible characters of the symmetric group [3, p.153]. In this connection, see also, for example, exercise 6, p.38, of [4]. We shall denote this Cauchy matrix, whose (i,j) entry is $(\alpha_i - \beta_j)^{-1}$, by $C(\alpha, \beta)$. The author has been intrigued with the problem of finding a suitable setting for the Cauchy matrix, analogous to the role of the Vandermonde matrix in polynomial theory. The purpose of this paper is to relate $C(\alpha, \beta)$ to the Vandermonde matrix and show how its determinant and inverse may be evaluated. Since starting this work, we have found that our formula for the inverse is given, in an older formulation, in Section 353 of [5]. The referee of this paper has also pointed out that M. J. Newell has given an approach to the Cauchy matrix on p. 347 of [6] that is rather similar to our presentation in this paper. Thus our findings are certainly not new, but we hope that this subject may be of interest to those who are not specialists in symmetric functions.

We continue to use the interpolation polynomials p_i, based on the points $\alpha_1, \dots, \alpha_n$, and introduce a corresponding family of interpolation polynomials q_i, based on the points $\beta_1, \dots, \beta_n$. Thus

$$q_i = \frac{q}{q'(\alpha_i)(x - \beta_i)} \text{ where } q = (x - \beta_1)\dots(x - \beta_n).$$

Since $\{q_1, \ldots, q_n\}$ is a basis for $\mathcal{P}$, there exist elements e_{ik} of K with

$$p_i = \sum_{k=1}^{n} e_{ik} q_k$$

for $1 \leq i \leq n$. Evaluating the polynomials on each side of the equation above at β_j, we obtain $e_{ij} = p_i(\beta_j)$. Recalling the definition of p_i, we see that

$$e_{ij} = \frac{-p(\beta_j)}{p'(\alpha_i)(\alpha_i - \beta_j)}.$$

Consequently, if $E = (e_{ij})$, we have the relation

$$D(\alpha_1, \ldots, \alpha_n)E = -C(\alpha, \beta)P(\beta_1, \ldots, \beta_n),$$

where $D(\alpha_1, \ldots, \alpha_n)$ and $P(\beta_1, \ldots, \beta_n)$ are the diagonal matrices whose diagonal entries are $p'(\alpha_1), \ldots, p'(\alpha_n)$ and $p(\beta_1), \ldots, p(\beta_n)$, respectively. Expressing the polynomials p_i and q_i in terms of powers of x, we have, say,

$$p_i = \sum_{k=1}^{n} a_{ik} x^{k-1}$$

and

$$q_i = \sum_{k=1}^{n} b_{ik} x^{k-1}$$

for $1 \leq i \leq n$. Our discussion earlier shows that if $A = (a_{ij})$ and $B = (b_{ij})$, then

$$A = V(\alpha_1, \ldots, \alpha_n)^{-1}, \quad B = V(\beta_1, \ldots, \beta_n)^{-1}.$$

However, we clearly have $A = EB$ and we obtain the relation

$$V(\alpha)^{-1} = -D(\alpha_1, \ldots, \alpha_n)^{-1}C(\alpha, \beta)P(\beta_1, \ldots, \beta_n)V(\beta)^{-1},$$

where we have written $V(\alpha)$ and $V(\beta)$ in place of $V(\alpha_1, \ldots, \alpha_n)$ and $V(\beta_1, \ldots, \beta_n)$. Thus we have proved the following result.

Theorem 1. *Let $\alpha_1, \ldots, \alpha_n$ be n different elements in K and let p be the polynomial*

$$(x - \alpha_1) \ldots (x - \alpha_n)$$

in $K[x]$. Let $\beta_1, \ldots, \beta_n$ be a further n different elements in K with $\alpha_i \neq \beta_j$ for all i and j. Let $C(\alpha, \beta)$ be the $n \times n$ Cauchy matrix whose (i, j) entry is $(\alpha_i - \beta_j)^{-1}$. Then we have the equation

$$C(\alpha, \beta) = -D(\alpha_1, \ldots, \alpha_n)V(\alpha)^{-1}V(\beta)P(\beta_1, \ldots, \beta_n)^{-1}.$$

Here $V(\alpha) = V(\alpha_1, \ldots, \alpha_n)$ and $V(\beta) = V(\beta_1, \ldots, \beta_n)$ are the Vandermonde matrices based on the α_i and β_j, respectively, and $D(\alpha_1, \ldots, \alpha_n)$ and $P(\beta_1, \ldots, \beta_n)$ are the $n \times n$ diagonal matrices whose diagonal entries are $p'(\alpha_1), \ldots, p'(\alpha_n)$ and $p(\beta_1), \ldots, p(\beta_n)$, respectively.

Corollary 1. *The determinant of the Cauchy matrix is*

$$(-1)^{n(n-1)/2} \frac{\prod_{i>j}(\alpha_i - \alpha_j) \prod_{i>j}(\beta_i - \beta_j)}{\prod_{i,j}(\alpha_i - \beta_j)}.$$

Proof. We may assume that the α_i and the β_j are all different, since otherwise the determinant is clearly 0 and the formula holds in this case. Theorem 1 shows that we have

$$\det C(\alpha, \beta) = (-1)^n \frac{\prod_{i=1}^n p'(\alpha_i) \prod_{i>j}(\beta_i - \beta_j)}{\prod_{i=1}^n p(\beta_i) \prod_{i>j}(\alpha_i - \alpha_j)}.$$

However, it is easy to verify that

$$\prod_{i=1}^{n} p'(\alpha_i) = (-1)^{n(n-1)/2} \prod_{i>j} (\alpha_i - \alpha_j)^2$$

and

$$\prod_{i=1}^{n} p(\beta_i) = (-1)^{n^2} \prod_{i,j} (\alpha_i - \beta_j)$$

and the result follows.

It is easy to find the inverse of the Cauchy matrix in the case that its determinant is non-zero. In Theorem 1, we write $V(\alpha)$, $V(\beta)$; $D(\alpha)$ and $P(\beta)$ in place of $V(\alpha_1,\ldots,\alpha_n)$, $V(\beta_1,\ldots,\beta_n)$, $D(\alpha_1,\ldots,\alpha_n)$ and $P(\beta_1,\ldots,\beta_n)$, respectively. Then Theorem 1 gives

$$C(\alpha,\beta) = -D(\alpha)V(\alpha)^{-1}V(\beta)P(\beta)^{-1}.$$

Interchanging the roles of the α_i and β_j, we have

$$C(\beta,\alpha) = -E(\beta)V(\beta)^{-1}V(\alpha)Q(\alpha)^{-1},$$

where $E(\beta)$ and $Q(\alpha)$ are the $n\times n$ diagonal matrices whose i-th diagonal entries are $q'(\beta_i)$ and $q(\alpha_i)$ for $1\le i\le n$, respectively. Assuming that $C(\alpha,\beta)$ is invertible, we obtain the relation

$$C(\alpha,\beta)^{-1} = P(\beta)E(\beta)^{-1}C(\beta,\alpha)Q(\alpha)D(\alpha)^{-1}.$$

We also observe that $C(\beta,\alpha) = -C(\alpha,\beta)'$, the prime denoting transpose. We have therefore proved the following result.

Theorem 2. *Let $\alpha_1,\ \ldots,\ \alpha_n$ and $\beta_1,\ \ldots,\ \beta_n$ be $2n$ different elements in K and let p and q be the polynomials*

$$(x-\alpha_1)\ldots(x-\alpha_n) \text{ and } (x-\beta_1)\ldots(x-\beta_n),$$

respectively. Then we have the relation

$$C(\alpha,\beta)^{-1} = -\Lambda_1 C(\alpha,\beta)'\Lambda_2,$$

where Λ_1 and Λ_2 are the diagonal matrices whose i-th diagonal entries are $p(\beta_i)/q'(\beta_i)$ and $q(\alpha_i)/p'(\alpha_i)$, respectively. In particular, the (i,j) entry of $C(\alpha,\beta)^{-1}$ is

$$\frac{p(\beta_i)q(\alpha_j)}{(\beta_i-\alpha_j)p'(\alpha_j)q'(\beta_i)}.$$

As an example of the use of this formula, we consider the case that $\alpha_i = i-1$ and $\beta_i = -i$ for $1\le i\le n$. The corresponding

Cauchy matrix based on these values is

$$\begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{3} & \cdots \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots \\ \vdots & \vdots & \vdots & \ddots \\ \frac{1}{n} & \frac{1}{n+1} & \frac{1}{n+2} & \cdots \end{pmatrix}.$$

This matrix is usually called a Hilbert matrix. The polynomials p and q for this matrix are

$$x(x-1)\ldots(x-n+1) \text{ and } (x+1)(x+2)\ldots(x+n).$$

Theorem 2 shows that the (i,j) entry of the inverse of the Hilbert matrix is

$$\frac{(-1)^{i+j}(n+i-1)!(n+j-1)!}{(i+j-1)(i-1)!^2(j-1)!^2(n-i)!(n-j)!},$$

which equals

$$(-1)^{i+j}(i+j-1)\binom{n+i-1}{n-j}\binom{n+j-1}{n-i}\binom{i+j-2}{i-1}^2,$$

as shown in [2, p.306].

References

[1] A. L. Cauchy, Exercices d'analyse et de phys. math. 2, (second edition). Bachelier: Paris, 1841.

[2] M.-D. Choi, *Tricks or treats with the Hilbert matrix*, Amer. Math. Monthly **90** (1983), 301–311.

[3] F. G. Frobenius, *Über die Charaktere der symmetrischen Gruppe, (1900)* in Gesammelte Abhandlungen, Band 3, Springer-Verlag: Berlin–Heidelberg–New York, 1968, 148–166.

[4] I. G. Macdonald, Symmetric functions and Hall polynomials. Clarendon Press: Oxford, 1979.

[5] T. Muir, A treatise on the theory of determinants, (revised and enlarged by W. H. Metzler). Dover Publications: New York, 1960 (reprint of 1933 original).

[6] M. L. Newell, *On the quotients of alternants and the symmetric group*, Proc. London Math. Soc. (2) **53** (1951), 345–355.

R. Gow,
Department of Mathematics,
University College,
Belfield,
Dublin 4.

SOME QUESTIONS CONCERNING THE VALENCE OF ANALYTIC FUNCTIONS

J. B. Twomey

In this short note we discuss, and illustrate by means of some examples, certain questions concerning the *valence* of analytic functions of one complex variable, that is, the number of times such functions take their values. We present a theorem which asserts the existence of certain constants relating to the valence of analytic functions in the unit disc, and conclude the note by raising some questions regarding these constants for the reader.

We begin with a definition. Suppose a function f is analytic in a domain D in the complex plane. We say that f is *p-valent* in D, p a positive integer, if (*i*) f takes no value more than p times in D, and (*ii*) f takes at least one value exactly p times in D. If $p = 1$ we have, of course, a *univalent* (or one-to-one) function. The following result for univalent functions is elementary and known:
(1) *If f is analytic in the unit disc $U = \{z : |z| < 1\}$ and univalent in the annulus*

$$A(\delta) \equiv \{z : \delta < |z| < 1\},$$

where $0 < \delta < 1$, then f is univalent in the full disc U.
This result is an easy consequence of Darboux's theorem [1, p. 115]: If f is analytic on and inside a simple closed curve γ, and f takes no value more than once on γ, then f is univalent inside γ.

It is natural to attempt to generalize (1) and to ask whether there is an analogous result for p-valent functions when $p > 1$. (This question was first posed by A. W. Goodman in a seminar in Tampa many years ago and this author's interest in these problems dates — albeit discontinuously — from that occasion.) We note immediately that the direct analogue of (1), namely

(2) *f analytic in U and p-valent in $A(\delta)$, $0 < \delta < 1$ and $p > 1$*
$\Rightarrow f$ is p-valent in U,

is false for every $p > 1$. We illustrate this here for the case $p = 2$ with an example which shows that, given any δ in $(0, 1)$, there exists a polynomial which is 2-valent in $A(\delta)$, but which is not 2-valent in U.

Example 1. *Let $P_n(z) = z(z^2 - \alpha_n)$, where $\alpha_n = 1 - 1/4n^2$. Then, for $n \geq 2$, P_n is 2-valent in the annulus $A(\frac{1}{n})$ and 3-valent in U.*

To understand this example — simple as it is — it is helpful to examine the image of the unit circle $C = \{z : |z| = 1\}$ under the mapping $w = P_n(z)$.

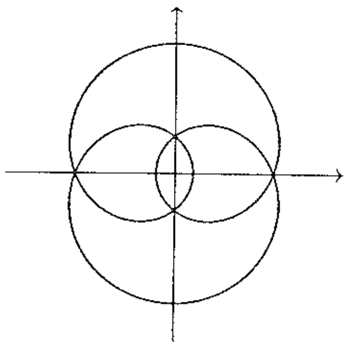


Fig. 1 $P_n(C)$

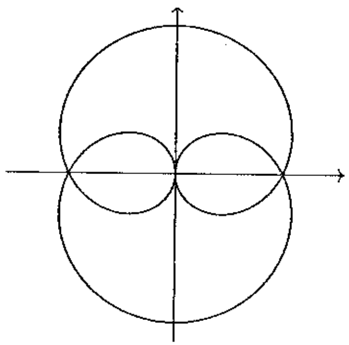


Fig. 2 $P(C)$

See Fig. 1. Now if β is any point inside the bounded component O_n of $E_n = \mathbf{C} \setminus P_n(C)$ that contains the origin, then

$$\frac{1}{2\pi}\Delta arg\{P_n(z) - \beta\} = 3,$$

where Δarg denotes the net change in the argument as z traverses C in the positive sense. Hence, by the *argument principle* [1, p. 104], every such value β is taken exactly three times in U by P_n. For similar reasons, every value in each of the other four bounded components of E_n is taken exactly once or twice only. The component O_n shrinks to the empty set as $n \to \infty$ (see Fig.2 for $P(C)$, where $P(z) = z^3 - z = \lim_{n\to\infty} P_n(z)$), and, as $P_n(0) = 0$, it is clear that, for each $n \geq 2$, there is a disc D_n centred at the origin with radius ε_n (decreasing to zero as $n \to \infty$) such that $P_n(D_n) \supset O_n$. But then P_n can take no value more than twice in $U \setminus D_n$ and (assuming that $U \setminus D_n$ contains the two non-zero zeros of P_n) is thus 2-valent in the annulus $A(\varepsilon_n)$. We leave it to the reader to *prove* that this is so with $\varepsilon_n = \frac{1}{n}$ for $n \geq 2$.

A function f satisfying the conditions in (2) is not necessarily p-valent in U, therefore, but it is the case (and easy to prove) that such a function is q-valent in U for *some* positive integer q. The value of q can be arbitrarily large, however. Indeed, as our next example shows, given $q \geq p \geq 2$, there exists an analytic function which is p-valent in $A(\delta)$, for some δ in $(0, 1)$, and q-valent in U.

Example 2. *Let p and q be integers with $q \geq p \geq 2$ and set $F(z) = exp(q\pi z)$. Then F is p-valent in the annulus*

$$\{z : \sqrt{1 - (\frac{4p-3}{4q})^2} < |z| < 1\}$$

(for instance), and q-valent in U.

This example, as the reader will readily verify, is an easy consequence of the standard periodicity property of the exponential function.

Example 2 leaves open the possibility that if f is any function satisfying the conditions in (2), and q is an integer greater than p, then f is at most q-valent in U, *provided δ is small enough.* This, finally, is indeed — with a qualification — essentially what our theorem asserts.

Theorem 3. *Suppose that p,q are integers with $p \geq 2$ and $q \geq 2p$, and that f is analytic in U. There exists a (largest) number*

$r^*(p,q)$ in $(0,1)$ such that if f is p-valent in an annulus $A(\delta)$, and $0 < \delta < r^*(p,q)$, then f is at most q-valent in U.

The author's proof of this result — which is based on a *normal family* [1, p. 213] argument, as a complex analyst reader might anticipate — is somewhat technical in detail and sheds no light on how the questions raised by the theorem might be answered, so we do not include it here. One question which arises is whether the theorem is true if we replace the condition '$q \geq 2p$' with '$q > p$', but a more fundamental question is :

What is the value of $r^*(p,q)$ for each permissible pair (p,q)?

We conclude by leaving these open questions, unclouded by any conjectures, for the reader.

Reference

[1] R. P. Boas, Invitation to Complex Analysis. Random House, 1987.

J. B. Twomey,
University College,
Cork.

ON A QUESTION POSED BY GRAHAM HIGMAN

Gerard M. Enright

Consider a function f of the non-negative integers given by the following rules:

$$f(3n) = 4n$$
$$f(3n+1) = 4n+1$$
$$f(3n+2) \text{ is undefined} \qquad (n = 0,1,2,3,\ldots).$$

Since $f(0) = 0$ and $f(1) = 1$, the function may be repeatedly and indefinitely applied to 0 and 1; that is, for $z = 0$ and 1, $f^k(z)$ is defined for all $k > 0$.

Question: Is there any integer $z > 2$ such that $f^k(z)$ is defined for all $k > 0$?

This function was introduced by Professor Graham Higman [1] during a lecture on explicit embeddings of finitely presented groups. He posed the question and he conjectured that the answer was "No". To be precise, he declared "No" to be his "first best guess".

In this paper, we will not prove Higman's conjecture but we will produce a good deal of evidence in its favour. Neither will we discuss the group theoretic context in which the question was raised. Instead we present an exploration of the problem as an example of computer-aided mathematics suitable for secondary school and college level students.

We use elementary programs in BASIC to obtain data on the function and we use this data in further development of the problem, leading to more efficient programming. Our suggestion is that students' knowledge and understanding of mathematics is reinforced by doing mathematics and that a computer is a very useful tool in this process. We demonstrate the power and scope of electronic computation. We also show its limitations when faced with a great volume of calculations and with very large numbers.

Let z be a positive integer. The sequence obtained by repeated application of the function f, $\{z, f(z), f^2(z), f^3(z), \ldots\}$, will be called the f-string of z. If z is such that $f^{(m-1)}(z)$ is congruent to 2 mod 3 for some m then $f^m(z)$ is undefined and the f-string of z, $\{z, f(z), f^2(z), f^3(z), \ldots, f^{(m-1)}(z)\}$, has length m. Otherwise z has infinitely long f-string and the question is whether any such $z > 2$ exists.

The following program may be used to compute f-strings.

```
10 INPUT S
30 PRINT S,;
40 IF(S MOD 3)=0 THEN S=4*S/3:GOTO 30
50 IF(S MOD 3)=1 THEN S=4*((S-1)/3)+1:GOTO 30
60 IF(S MOD 3)=2 THEN PRINT"STOP"
80 END
```

Here are two results:

```
7    9    12   16   21   28   37   49   65   STOP
19   25   33   44   STOP
```

Here are two more in f-string notation:

$\{264, 352, 469, 625, 833\}$

$\{961, 1281, 1708, 2277, 3036, 4048, 5397, 7196\}$.

Our first objective is to prove that there is no integer z between 2 and 1,000 for which $f^k(z)$ is defined for all $k > 0$. We do this by computing the f-strings of all such z. Our use of this method betrays the fact that we expect all f-strings to be finite, as we are inclined to support Higman's conjecture. We then consider the feasibility of extending our methods to a higher number range.

The program above may be suitably amended by the following additions:

```
10 FOR T=2 TO 1000
20 S=T
70 NEXT T
```


Running on a BBC Master 128 microcomputer all f-strings are screened in 51 seconds and transferred to paper on a Star LC-10 dot-matrix printer in 10.6 minutes. Program line 60 is not necessary but it is useful, on a large printout of numbers, to highlight where each f-string stops.

Students may be patient enough to wait 10.6 minutes for a listing of the first 1,000 f-strings but it is unlikely that either they or their teachers will want to extend this method much further. It would take two class periods to list 10,000 and more than a week of non-stop running to list all f-strings up to 1 million. Surely we can do better than that.

All integers in the sequence $2, 5, 8, 11, 14, \ldots$ have f-strings of length 1 simply because $f(3n+2)$ is undefined. So let's not waste time checking them. The following program is a little better.

```
10 FOR T=3 TO 999 STEP 3
20 FOR R=T TO T+1
25 S=R
30 PRINT S,;
40 IF(S MOD 3)=0 THEN S=4*S/3:GOTO 30
50 IF(S MOD 3)=1 THEN S=4*((S-1)/3)+1:GOTO 30
60 IF(S MOD 3)=2 THEN PRINT"STOP"
65 NEXT R
70 NEXT T
80 END
```

With this program it takes 45 seconds to screen the f-strings and 7.8 minutes to print them out.

Two sequences of numbers have f-strings of length 2. These are:

$4, 13, 22, 31, 40, \ldots$

$6, 15, 24, 33, 42, \ldots,$

the sequences $\{9n + 4 : n = 0, 1, 2, 3, \ldots\}$ and $\{9n + 6 : n = 0, 1, 2, 3, \ldots\}$ respectively.

We are prompted to look at the function definition with integers written in terms of their least residues mod 9 rather than mod 3. f is undefined on all integers of the form $9n + 2$, $9n + 5$ and $9n + 8$. f^2 is undefined in two cases since $f(9n+4) = 12n+5$ and $f(9n + 6) = 12n + 8$. Hence, 5 out of every 9 integers have

f-strings of lengths 1 or 2 and we need only examine the other 4. The revised program screens results in 37 seconds and prints them in 6.1 minutes.

From the pattern of STOPS indicating f-strings of length 3 we are led to examine the function definition with integers expressed in terms of least residues mod 27. The following results are easily established.

f is undefined on all integers of the following forms:
$27n+\ 2,\ 27n+\ 5,\ 27n+\ 8,\ 27n+11,\ 27n+14,$
$27n+17,\ 27n+20,\ 27n+23,\ 27n+26.$

f^2 is undefined in the following situations:

$$\begin{aligned} f(27n+4) &= 36n+5, \\ f(27n+6) &= 36n+8, \\ f(27n+13) &= 36n+17, \\ f(27n+15) &= 36n+20, \\ f(27n+22) &= 36n+29, \\ f(27n+24) &= 36n+32. \end{aligned}$$

f^3 is undefined in the following situations:

$$\begin{aligned} f^2(27n+\ 3) &= f(36n+\ 4) = 48n+5, \\ f^2(27n+10) &= f(36n+13) = 48n+17, \\ f^2(27n+18) &= f(36n+24) = 48n+32, \\ f^2(27n+25) &= f(36n+33) = 48n+44. \end{aligned}$$

With this analysis we may confine attention to just 8 out of every 27 numbers and we know that all excluded integers have f-strings of length at most 3. The following is our best program so far.

```
10 FOR T=7 TO 979 STEP 27
20 FOR R=1 TO 8
30 ON R GOTO 40,45,50,55,60,65,70,75
40 S=T:GOTO 80
45 S=T+2:GOTO 80
```



```
 50 S=T+5:GOTO 80
 55 S=T+9:GOTO 80
 60 S=T+12:GOTO 80
 65 S=T+14:GOTO 80
 70 S=T+20:GOTO 80
 75 S=T+21
 80 PRINT S,;
 90 IF(S MOD 3)=0 THEN S=4*S/3:GOTO 80
100 IF(S MOD 3)=1 THEN S=4*((S-1)/3)+1:GOTO 80
110 IF(S MOD 3)=2 THEN PRINT"STOP"
120 NEXT R
130 NEXT T
140 END
```

A screen run takes 30 seconds and printout time is 4.5 minutes. We have reduced running time by 40% and printing time by nearly 60%. It would still take 3 days of non-stop running to reach our 1 million target.

To continue with this approach, we should now analyse function behaviour with integers written in terms of least residues mod 81. We should exclude from testing all integers which have f-strings of length at most 4. The work involved might be considered rather cumbersome.

Alternatively, we may further exploit the fact that the only integers with f-strings of length 4 or more are those of form $27n+$ one of $\{0,1,7,9,12,16,19,21\}$. The general form of $f^3(z)$ may be calculated in each case. The values are $64n+$ the corresponding element of $\{0,1,16,21,28,37,44,49\}$. Let us restrict attention to integers of this latter form. Eliminate all integers congruent to 2 mod 3, all congruent to 4 or 6 mod 9 and all congruent to 3, 10, 18 or 25 mod 27. The f-strings produced, when the revised program is run through the range $16=f^3(7)$ to $2369=f^3(1000)$, are of length at least 7 but the first 3 elements of each one are omitted. A screen run takes 12.5 seconds and paper printout takes 1.5 minutes. Extension of this method might also be considered cumbersome.

The idea of restricting output to the longer f-strings may be used in a more elegant process. Let us modify the program listed above so as to output only those f-strings of length at least 7. The new listing is as follows:

```
 10 DIM F(7)
 20 FOR T=7 TO 979 STEP 27
 30 FOR R=1 TO 8
 40 ON R GOTO 50,60,70,80,90,100,110,120
 50 S=T:GOTO 130
 60 S=T+2:GOTO 130
 70 S=T+5:GOTO 130
 80 S=T+9:GOTO 130
 90 S=T+12:GOTO 130
100 S=T+14:GOTO 130
110 S=T+20:GOTO 130
120 S=T+21
130 F(0)=S
140 FOR I=1 TO 6
150 IF (S MOD 3)=0 THEN S=4*S/3:GOTO 180
160 IF (S MOD 3)=1 THEN S=4*((S-1)/3)+1:GOTO 180
170 IF (S MOD 3)=2 THEN 250
180 F(I)=S
190 NEXT I
200 FOR J=0 TO 6:PRINT F(J),;:NEXT J
210 IF (S MOD 3)=0 THEN S=4*S/3:PRINT S,;:GOTO 210
220 IF (S MOD 3)=1 THEN S=4*((S-1)/3)+1:PRINT S,;:GOTO
                                                    210
230 IF (S MOD 3)=2 THEN PRINT "STOP"
240 PRINT:PRINT
250 NEXT R
260 NEXT T
270 END
```

On a run of this program, the 88 results are printed in 2.2 minutes. A re-run with the following changes:

```
 10 DIM F(13)
140 FOR I=1 TO 12
200 FOR J=0 TO 12:PRINT F(J),;:NEXT J
```

yields the 8 integers between 2 and 1,000 which have f-strings of length at least 13 in just 18 seconds. The longest f-strings in

this range are those of 163 and 331. Both of these have length 15 and they end with $f^{14}(163) = 9104$ and $f^{14}(331) = 18545$. We have a procedure now which we may reasonably hope to apply to higher numbers. Students might be encouraged to find out how many integers up to 10,000 have f-strings of length 19 or more and which of these has the longest f-string. How many integers up to 100,000 have f-strings of length 25 or more? Which of these is the longest?

The program is easily amended for these investigations. For example:

```
 10 DIM F(19)
 20 FOR T=7 TO 9997 STEP 27
140 FOR I=1 TO 18
200 FOR J=0 TO 18:PRINT F(J),;:NEXT J
```

A run of this modified program shows that there are just 5 integers up to 10,000 with f-strings of length at least 19. Output takes 2.8 minutes. The numbers are: $3475, 4633, 6177, 8236, 8607$.

The first of these, 3475, has the longest f-string. It ends with

$$f^{23}(z) = 2596901.$$

The next three f-strings share this endpoint because

$$f(3475) = 4633, f(4633) = 6177, f(6177) = 8236.$$

Let us go a step further with the following changes:

```
 10 DIM F(25)
 20 FOR T=7 TO 49984 STEP 27
140 FOR I=1 TO 24
200 FOR J=0 TO 24:PRINT F(J),;:NEXT J
```

A run now takes 14.85 minutes. There are just 5 integers up to 50,000 with f-strings of length at least 25. The longest is that of 38,754, which ends with

$$f^{27}(38754) = 91549952.$$

Extending the range to 100,000 yields another 7 integers with f-strings of length 25 or more. There are 6 such integers between

100,000 and 200,000 and 6 more between 200,000 and 300,000. So far then, we have shown that there are just 24 integers z in the range 2 to 300,000 for which $f^{24}(z)$ is defined. The integer 38,754 still has one of the longest f-strings at 28 terms, a length equalled only by 65,610 and not exceeded.

Having reached 300,000 without finding an infinite f-string, we are inclined to rush onwards but we face two problems. We are approaching accuracy limits of the computer language (BBC BASIC) and program running times are rather slow (for classwork).

On the question of time, our current program, which is designed to list f-strings of length 25 or more, takes about 28 minutes for each 100,000 number range. Our 1 million time estimate stands at 4 hours and 40 minutes. We would rather not sacrifice program simplicity and legibility for minor efficiencies but one significant improvement would be to change all variables to integer type. Also integer division is executed faster than ordinary division. With these alterations, the program runs on a BBC Master 128 at about 22 minutes for each 100,000.

Schools and colleges will also have other equipment. Comparisons may be made of running times of similar programs on different computers, in different versions of BASIC and in other languages. The author also used an RM Nimbus X20. This 80186 based 8MHz computer supports BBC BASIC and runs it faster than the BBC Master Series. It is not necessary to use integer variables to take advantage of quicker integer calculations. The Nimbus runs our current program about 40% faster. The printer used in this experiment was an Epson LQ800 but printer speeds are not very significant now that output volume is considerably reduced.

Whether or not one is satisfied with this time, roundoff errors will ruin any attempt to go further. The next integer for which $f^{24}(z)$ is defined is $z = 335167$. Either computer will accurately produce the f-string of z up to

$$f^{27}(z) = 791783233$$

but then give 1.05571098 E9 and the message "Too big at line 240". Now the student can have the satisfaction of doing a few simple divisions and multiplications with pen and paper to produce the rest of the f-string beyond the capability of the machines. Results are as follows:

$$f^{28}(z) = 1055710977$$
$$f^{29}(z) = 1407614636$$
$$f^{30}(z) \text{ is not defined.}$$

We might amend the program by inserting brackets to give division priority over multiplication in the calculation of successive function values in the variable S. This however merely postpones the inevitable breakdown to $z = 491731$. The maximum integer which can be handled by BBC BASIC is 2,147,483,647. Larger real numbers can be stored but tenth and subsequent digits will be rounded and accuracy lost soon after the maximum integer value. Again the student who is not afraid of a few long calculations can go beyond the computer for the rest of the f-string of 491,731. It ends at a length of 40 terms with

$$f^{39}(491731) = 36672278528.$$

The student who survives that calculation will not want to give up before reaching the 1 million target. Would you be prepared to omit those program lines which compute the 26th and subsequent terms of the long f-strings? Let the computer stop at the 25th term and thus avoid roundoff errors and numbers which are too big. The Nimbus works at about 13.5 minutes for each 100,000 number range or 2 hours 15 minutes for a million run and there are a total of 69 integers z between 2 and 1,000,000 for which $f^{24}(z)$ is defined.

Since the Nimbus can handle real numbers which are just a little bigger than the maximum integer. some more help may be squeezed from the computer. The program can be extended successfully to give 28 terms of sufficiently long f-strings. There are just 17 survivors and it seems reasonable to complete that number of calculations by hand. The results are as follows:

Seven integers less than 1,000,000 have f-strings of length 28:

38754	65610	563401	595852
725097	972979	988666.	

Four integers have f-strings of length 29:

422551	446889	543823	827199.

Three integers have f-strings of length 30:

335167	794422	940402.

The three remaining integers are:

491731	655641	874188.

The f-strings of these three have the exceptional lengths of 40, 39 and 38 respectively. All end with the same number 36,672,278,528. In fact, the last two are substrings of the first because

$$f(491731) = 655641 \quad f(655641) = 874188.$$

More powerful personal computers in the 80286 and 80386 ranges may be available to some students. Microsoft GW BASIC is normally supplied in the MS-DOS package. Greater accuracy and faster running times can be achieved. The author transferred the program to a 25 MHz Morse 486 personal system. Double precision numbers have an accuracy level of 17 digits internally with up to 16 displayed. The MOD operator, however, so useful for modulus arithmetic, has an upper limit of 32,767 and must be replaced by direct computation. A million run can be achieved, in a time of 24.4 minutes, on the Morse 486, printing full f-strings for all integers z between 2 and 1,000,000 for which $f^{24}(z)$ is defined.

It is observed that the hand completed results are all verified by the machine and it is felt that this is a good point at which to end the article. The patient reader might, however, like to know that the f-string of length 40 arising from 491,731 is not only the longest of any integer less than 1 million but it is also the longest f-string of any integer less than 10 million. Soon after that, it is equalled and then surpassed. The longest f-string of any integer less than 100 million is that of 95,305,399 which has 47 terms.

Reference

[1] G. Higman, *Some explicit embeddings of finitely presented groups* (Lecture at Groups in Galway Conference, May 1990) (Unpublished).

Gerard M. Enright,
Department of Mathematics and Computer Studies,
Mary Immaculate College,
Limerick.

Book Review

QUADRATIC AND HERMITIAN FORMS OVER RINGS

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Max-Albert Knus.
Springer-Verlag, 1991,
ISBN 3 540 52117 8

Reviewed by David W. Lewis

The theory of quadratic forms was traditionally regarded as a part of number theory until the work of Witt in the 1930's. His work paved the way for the algebraic theory of quadratic forms over arbitrary fields, a branch of algebra involving a mixture of linear algebra, ring theory and field theory. Witt's work lay more or less dormant until the 1960's, when the work of Pfister demonstrated that there was a rich theory to be explored, and from there the subject really took off. Developments in topology (calculation of surgery obstruction groups), algebraic K-theory, and algebraic geometry led some mathematicians in the 1960's and onwards to examine quadratic and hermitian forms over various kinds of rings. Before that, the only work on forms over rings was of a number-theoretic nature, involving rings of integers.

This book introduces the reader to the theory of quadratic forms over commutative rings in a general setting. The author, M.-A. Knus, has been one of the principal researchers in this area over the last two decades or more. The book is suitable for graduate students, and for mathematicians working in other areas who wish to learn something of the subject. The reader is assumed to have a knowledge of the usual basic results in algebra, including some homological algebra. Unproved theorems are always quoted, unless they are basic results. For the latter part of the book, a

familiarity with some algebraic K-theory and algebraic geometry is helpful.

Chapter 1 introduces the basic definitions and terminology of forms, and develops tools which are used later. Chapter 2 deals with the general theory of forms in categories. There is considerable overlap here with Chapter 7 of the book of Scharlau [1]. Chapter 3 is entitled "Descent theory and cohomology". It introduces the technique of faithfully flat descent, and the notion of twisted forms. Chapter 4 lays the foundations of the theory of Clifford algebras for quadratic forms over rings. The Clifford algebra is used to define the discriminant, the Arf invariant, and the Witt invariant of a quadratic space. Chapter 5 describes the classification of quadratic spaces of low rank (specifically rank ≤ 6) via invariants such as the above. An interesting and surprising by-product of the work on forms of rank 6 over arbitrary commutative rings is a result about involutions of orthogonal type on rank 16 Azumaya algebras. A criterion for the decomposability of such an involution is obtained, utilizing an invariant called a Pfaffian discriminant. The involution decomposes if and only if the Pfaffian is trivial. The surprising thing about this result is that it was not observed at all in the special case of algebras over fields. Thus the more general setting of rings can sometimes yield results that have passed unnoticed for fields. Chapter 6 contains splitting, stability and cancellation theorems for unitary spaces. These are unitary versions of theorems of Bass, Serre and Vaserstein in algebraic K-theory, and are quite technical. Chapter 7 deals with polynomial rings, and is again fairly technical, utilizing some of the results of the previous chapter. Finally, Chapter 8 is concerned with the calculation of Witt groups of real affine curves and surfaces, an area in which there is currently a lot of research going on.

The book is well organized, clearly written, and seems to have few typographical errors. It is a welcome addition to the literature, and I warmly recommend it to those who wish to learn more about the general theory of quadratic and hermitian forms over rings.

Reference

[1] W. Scharlau, Quadratic and Hermitian Forms, (Grundlehren der mathematischen Wissenschaften 270). Springer-Verlag: Berlin, 1985.

David W. Lewis,
Department of Mathematics,
University College,
Dublin.

Book Review

CLASSICAL CHARGED PARTICLES

(Advanced Book Classics)

F. Rohrlich

Addison-Wesley, 1990, 305pp.
ISBN 0 201 51501 6

Reviewed by László Fehér

This book was originally published in 1965 as part of the Addison-Wesley Series in Advanced Physics. It is quite a unique text on the fundamental-theoretical aspects of the classical theory of charged particles. The author pays special attention to the logical structure of the subject and to properly placing it in the net of the bordering physical theories, such as special and general relativity, classical and quantum mechanics and quantum electrodynamics. The student, or indeed the researcher, has much to gain from the lucid exposition of the general structure of physical theory offered in this book through an example. The historical and philosophical aspects are also exhibited as an integral part of the theory.

The book consists of nine chapters, the first three of which deal with the philosophical and historical aspects of its subject matter and with the foundations of classical mechanics. Chapter 4 gives a detailed exposition of the Maxwell-Lorentz field equations, their solutions and symmetry properties, which form the basis for treating the theory of electromagnetic radiation in the next chapter. The central part of the book is Chapter 6, which deals with the equation of motion of the charged, classical elementary particle, given by the Lorentz-Dirac equation together with the asymptotic conditions. The derivation of the equation of motion on the basis of the Maxwell-Lorentz equations and the conservation laws, its mathematical properties, special solutions and the questions related to its physical interpretation, as well as the underlying action principle are treated here in detail. Chapter 7 is devoted to various generalizations of the equation of motion and the last two chapters explain the theory's relation with the other levels of physical theory, and summarize its principles and structure. There are also two appendices on the space-time of special and general relativity.

As set out in the Preface, in this book the author's purpose has been to demonstrate that with modern knowledge it is possible to complete the works of such men as Lorentz, Abraham, Poincaré and Dirac on the classical theory of charged particles and to show that the resultant structure is consistent and beautiful. His masterly exposition of the subject is very enjoyable to read. The publisher meets public demand by its present reissue as a volume in the Advanced Book Classics series.

László Fehér,
Université de Montréal,
Québec,
Canada.

INSTRUCTIONS TO AUTHORS

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