100 YEARS OF DIXON'S IDENTITY

James Ward

Although Queen's College, Galway cannot boast of such a world-class mathematician as Boole (Queen's College, Cork), nevertheless if one considers the size of Galway, the relative poverty of the hinterland and the remoteness of Galway from other centres of learning, mathematicians such as Allman, Dixon and Bromwich who each held the chair of mathematics at QCG were of a very high calibre indeed.

Perhaps the most distinguished of this trio was Dixon (though Bromwich would have his admirers too) who was appointed to the Chair in Mathematics at Queen's College, Galway in 1893 and to the Chair in Mathematics at Queen's College, Belfast in 1901, where he remained until his retirement in 1930. A broad account of Dixon's life and work can be found in the obituary written by E. T. Whittaker [6] from which the following biographical information has been extracted.

Alfred Cardew Dixon was born on the 23rd May, 1865 at Northallerton, Yorkshire, went up to Trinity College, Cambridge as a major scholar and in the Tripos of 1886 ("an exceptionally strong year" — Whittaker [6]) graduated as Senior Wrangler. He was awarded a Smith's Prize in 1888 and elected a fellow of Trinity College in the same year. In [6] Whittaker notes that in his early years Dixon had produced comparatively little work of real distinction, but that from 1893 (the year he was elected to the Chair in Mathematics at Queen's College, Galway) "he became a most productive original worker".

Dixon produced important memoirs on ordinary and partial differential equations, Abelian integrals, automorphic functions, Fredholm theory and functional equations. He was elected to

Fellowship of the Royal Society in 1904 and was President of the London Mathematical Society in 1931–33 following his retirement from Queen's University, Belfast. Dixon died on 4th May 1936 and had been predeceased by his wife in 1926; they had no children.

1991 marks the 100th anniversary of the appearance of a note by Dixon [1] proving the combinatorial identity

$$(*) \qquad \sum_{k=0}^{n}(-1)^k \left({}^nC_k\right)^3 = \begin{cases} \dfrac{(-1)^m(3m)!}{(m!)^3} & \text{if } n = 2m \\ 0 & \text{otherwise.} \end{cases}$$

In the literature on combinatorial theory this identity or the following slight generalisation due to Fjeldsted [3] (see also Dixon [2]) now bears Dixon's name:

$$\sum_k \binom{a+b}{a+k}\binom{b+c}{b+k}\binom{c+a}{c+k}(-1)^k = \frac{(a+b+c)!}{a!\,b!\,c!}$$

for integers a, b, $c \geq 0$ and the permitted range of the integer k (which is finite).

There are several proofs of Dixon's identity and I wish to present three such proofs, namely Dixon's original proof, a second using the Lagrange inversion formula as described in [4] and a third using WZ pairs [7, p.126]. These illustrate the rich diversity of techniques in Combinatorics, but I would also like to draw attention to the fine mathematical heritage of Queen's (now University) College, Galway as represented in the work of Allman, Bromwich, Dixon and others.

First Proof (Dixon [1])

For n a positive integer, and writing $(1+x)^n$ as

$$a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0$$

(here the a_i are the binomial coefficients ${}^nC_i$) it follows that

$$(**) \qquad a_0^3 - a_1^3 + a_2^3 - \ldots - a_n^3$$

is zero if n is odd, s an arbitrary positive integer (${}^nC_i = {}^nC_{n-i}$), so the question is to find a closed form for such a sum (∗∗) when n is even. When n is even, say $n = 2m$, and $s = 3$, we denote by S the alternating sum (∗∗) so obtained, and in this case S is the left hand side of the formula (∗). Now S is the coefficient of $y^{4m}z^{4m}$ in $(1-y^2)^{2m}(1-z^2)^{2m}(1-y^2z^2)^{2m}$, which is the term independent of y and z in

$$(y-y^{-1})^{2m}(z-z^{-1})^{2m}(yz-y^{-1}z^{-1})^{2m}.$$

Making the trigonometric substitutions $y = \cos\theta + i\sin\theta$ and $z = \cos\phi + i\sin\phi$, S becomes the absolute term in

$$(-4)^{3m}\sin^{2m}\theta\sin^{2m}\phi\sin^{2m}(\theta+\phi)$$

when expanded in cosines of multiples and sums of multiples of θ and ϕ, so (1)

$$(-1)^m 4^{-3m} 4\pi^2 S = \int_0^{2\pi}\int_0^{2\pi} \sin^{2m}\theta\,\sin^{2m}\phi\,\sin^{2m}(\theta+\phi)\,d\theta\,d\phi.$$

The Binomial Theorem is used to expand $\sin^{2m}(\theta+\phi)$ via $(\sin\theta\cos\phi + \cos\theta\sin\phi)^{2m}$, every second term in the integral vanishes and the right hand side of (1) reduces to

$$\sum_{k=0}^{m} {}^{2m}C_{2k} \int_0^{2\pi} \sin^{2(m+k)}\theta\,\cos^{2(m-k)}\theta\,d\theta \cdot \int_0^{2\pi} \sin^{2(2m-k)}\phi\,\cos^{2k}\phi\,d\phi.$$

Each integral is 4 times the integral over 0 to $\pi/2$. A further substitution of $x = \sin^2\theta$, $y = \sin^2\phi$ results in an integral of the type

$$\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,dx \cdot \int_0^1 y^{\gamma-1}(1-y)^{\delta-1}\,dy$$

which is a product of Beta integrals, equal to $\dfrac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}\cdot\dfrac{\Gamma(\gamma)\Gamma(\delta)}{\Gamma(\gamma+\delta)}$.
Using $\Gamma(\alpha) = (\alpha-1)\Gamma(\alpha-1)$, and $\Gamma(1/2) = \sqrt{\pi}$, and some diligent calculations, the $4\pi^2$ term in the left hand side of (1) cancels and we are left with

$$\sum_{k=0}^{m} \frac{(2m)!}{(2k)!\,(2m-2k)!} \times \frac{1\cdot3\cdot5\cdots(2m+2k-1)\cdot1\cdot3\cdot5\cdots(2m-2k-1)}{2\cdot4\cdot6\cdots4m} \times \frac{1\cdot3\cdot5\cdots(4m-2k-1)\cdot1\cdot3\cdot5\cdots(2k-1)}{2\cdot4\cdot6\cdots4m}$$

on the right hand side of (1), which simplifies (!) to

$$\frac{(4m)!}{2^{8m}(m!)^2(2m)!}\left\{1 + m\,\frac{2m+1}{4m-1} + \frac{m(m-1)}{2}\,\frac{(2m+1)(2m+3)}{(4m-1)(4m-3)} + \cdots \quad (m+1 \text{ terms})\right\} \qquad (2)$$

Dixon now refers to Wolstenholme's Problems 2nd edition #303 wherein

$$1 - m\,\frac{a}{b} + \frac{m(m-1)}{2!}\,\frac{a(a-1)}{b(b-1)} - \frac{m(m-1)(m-2)}{3!}\,\frac{a(a-1)(a-2)}{b(b-1)(b-2)} + \cdots \quad (m+1 \text{ terms})$$

is equal to

$$\left(1-\frac{a}{b}\right)\left(1-\frac{a}{b-1}\right)\left(1-\frac{a}{b-2}\right)\cdots\left(1-\frac{a}{b-m+1}\right)$$

and (2) falls into this pattern on putting $a = -m - 1/2$ and $b = 2m - 1/2$ which finally results in (∗).

Perhaps this clever proof justifies Whittaker's remark [6] "if the method was possible Dixon would make it work".

Second Proof (Goulden–Jackson [4, p.23–4])

This requires some preliminary notation. We denote by $R[[t]]$ the ring of formal power series in a variable t over a commutative ring

R, so an element f of $R[[t]]$ is of the form $f = \sum_{k\geq 0} a_k t^k$, $a_k \in R$. If we allow finitely many negative powers of t, we have the ring of (formal) Laurent series $R((t))$. A polynomial is an element of $R[[t]]$ with only finitely many non-zero a_k. Two important subsets of $R[[t]]$ are

$$R[[t]]_0 = \{f \in R[[t]] \mid a_0 = 0\} \quad \text{and}$$
$$R[[t]]_1 = \{f \in R[[t]] \mid a_0^{-1} \text{ exists}\};$$

in this latter case f has an inverse in $R[[t]]$ (long division!). We let f' denote the formal derivative of f which is $\sum_{k\geq 0}(k+1)a_{k+1}t^k$, and define the "coefficient operator" $[t^k]f$ to pick out the coefficient of t^k in f, thus $[t^k]f = a_k$ and $[t^k]f' = (k+1)a_{k+1}$. Finally if $f_1 \in R((t))$, the valuation of f_1 is defined to be

$$\operatorname{val}(f_1) = \begin{cases} k & \text{if } f_1(t) = t^k g(t),\ g(t) \in R[[t]]_1 \\ \infty & \text{otherwise.} \end{cases}$$

Now we can state Lagrange's Inversion Formula. Let $\phi(t) \in R[[t]]_1$. Then there exists a unique formal power series $w(t) \in R[[t]]_0$ such that $w(t) = t\phi(w(t))$. Moreover if $f(x) \in R((x))$ (Laurent series) then (a)

$$[t^n]f(w) = \begin{cases} \dfrac{1}{n}[x^{n-1}]\{f'(x)\phi(x)^n\} & \text{for } n \neq 0,\ n \geq \operatorname{val}(f) \\ [x^0]f(x) + [x^{-1}]f'(x)\log(\phi(x)\phi(0)^{-1}) & \text{for } n = 0 \end{cases}$$

$$\left[\exp(x) = \sum_{j\geq 0}\frac{x^j}{j!} \in R[[x]], \quad \text{and} \quad \log(\exp(x)) = x\right].$$

If $F(x) \in R[[x]]$ then (b)

$$\sum_{n\geq 0} c_n t^n = F(w)\{1 - t\phi'(w)\}^{-1} \quad \text{where } c_n = [x^n]\{F(x)\phi(x)^n\}.$$

Example

Suppose we want to invert $w(t) = te^w$, ie to express w as a power series in t, we have $\phi(t) = e^t$, $f(w) = w$, and $\operatorname{val}(f) = 1$; so by (a)

$$[t^0]w(t) = [x^0]f(x) + [x^{-1}]1 \cdot \log(e^x \cdot 1) = 0$$

and $$[t^n]w(t) = \frac{1}{n}[x^{n-1}]\{1 \cdot e^{nx}\} = \frac{1}{n}[x^{n-1}]\left\{\sum_{k\geq 0}\frac{(nx)^k}{k!}\right\}$$
$$= \frac{1}{n}\frac{n^{n-1}}{(n-1)!} = \frac{n^{n-1}}{n!}$$

hence $w(t) = \sum_{n\geq 1}(n^{n-1}/n!)\,t^n$, note that $w(t) \in R[[t]]_0$. Lagrange's formula will also give power series expansions for $w^{-1}(t)$ $(f(w) = w^{-1})$ or $w^{-2}(t)$ $(f(w) = w^{-2})$ etc. ■

There is a multivariate version of the Lagrange formula [4, p.21] and the multivariate version of (b) reads: If

$$F(\mathbf{x}),\ \phi_1(\mathbf{x}),\ \ldots,\ \phi_m(\mathbf{x}) \in R[[\mathbf{x}]] \quad \text{where } \mathbf{x} = (x_1, \ldots, x_m),$$

and if

$$w_i = t_i\phi_i(\mathbf{w}) \text{ for } 1 \leq i \leq m \quad \text{where } \mathbf{w} = (w_1, \ldots, w_m),$$

then

$$\left.\frac{F(\mathbf{w})}{\det\left(\delta_{ij} - t_i\partial\phi_i(\mathbf{w})/\partial w_j\right)}\right|_{\mathbf{w}=\mathbf{w}(t)} = \sum_{\mathbf{k}\geq \mathbf{0}} \mathbf{t}^{\mathbf{k}}[\mathbf{x}^{\mathbf{k}}]\{F(\mathbf{x})\phi(\mathbf{x})^{\mathbf{k}}\}$$

where $\mathbf{k} = (k_1, \ldots, k_m)$.

For instance suppose we have $X_i = \sum_{j=1}^m a_{ij}x_j$ and we want to find the coefficient of $\mathbf{x}^{\mathbf{k}}$ in $\mathbf{X}^{\mathbf{k}}$, ie the coefficient of $x_1^{k_1}\ldots x_m^{k_m}$ in $X_1^{k_1}\ldots X_m^{k_m}$. We apply the above with $F(\mathbf{x}) = 1$, $\phi_i(\mathbf{x}) = a_{i1}x_1 + \ldots + a_{im}x_m$ so the ϕ_i are linear functions, to get that the coefficient of $\mathbf{x}^{\mathbf{k}}$ in $\mathbf{X}^{\mathbf{k}}$ is equal to the coefficient of $\mathbf{x}^{\mathbf{k}}$ in $|I - x_i a_{ij}|^{-1}$ where I is the $m \times m$ identity matrix. If we put $A = (a_{ij})$ and $X = \operatorname{diag}(x_1, \ldots, x_m)$ the formula reads as the coefficient of $x_1^{k_1}\ldots x_m^{k_m}$ in $|I - XA|^{-1}$. This specific result for linear ϕ_i is the so-called MacMahon Master Theorem. To apply this to Dixon's problem of evaluating S (**) we note that

$$\left(1-\frac{x}{y}\right)^n\left(1-\frac{y}{z}\right)^n\left(1-\frac{z}{x}\right)^n$$
$$= \sum_{0\leq i,j,k\leq n}(-1)^{i+j+k}\,{}^nC_i\,{}^nC_j\,{}^nC_k\,x^{i-k}y^{j-i}z^{k-j}.$$

The term independent of x, y, z is when $i = j = k$, which is S. Thus

$$S = [x^0y^0z^0]\left\{\left(1-\frac{x}{y}\right)\left(1-\frac{y}{z}\right)\left(1-\frac{z}{x}\right)\right\}^n.$$
$$= [x^ny^nz^n]\{(y-x)(z-y)(x-z)\}^n$$

Now $m = 3$, $\phi_1 = z - y$, $\phi_2 = x - z$, $\phi_3 = y - x$, and

$$A = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix} = \left(\frac{\partial \phi_i}{\partial x_j}\right)$$

where $(x_1, x_2, x_3) = (x, y, z)$. Then $S = [x^ny^nz^n]\left\{|I - XA|^{-1}\right\}$. Now

$$|I - XA| = \begin{vmatrix} 1 & x & -x \\ -y & 1 & y \\ z & -z & 1 \end{vmatrix} = 1 + yz + xy + xyz - xyz + xz$$
$$= 1 + xy + yz + zx,$$

so $S = [x^ny^nz^n](1 + xy + yz + zx)^{-1}$

$$= [x^ny^nz^n] \sum_{\alpha,\beta,\gamma\ge 0} (-1)^{\alpha+\beta+\gamma}\frac{(\alpha+\beta+\gamma)!}{\alpha!\,\beta!\,\gamma!}x^{\alpha+\gamma}y^{\alpha+\beta}z^{\beta+\gamma}.$$

Therefore we must have $\alpha+\gamma = \alpha+\beta = \beta+\gamma = n$ or $\alpha = \beta = \gamma = n/2$. However α, β, γ are integers, which fact forces $n = 2m$ say, and $\alpha+\beta+\gamma = 3m$. So $S = \dfrac{(-1)^m(3m)!}{(m!)^3}$ as required.

Third Proof

This uses the method of WZ pairs [7, p.120ff]. The idea of a WZ pair is as follows: To prove the identity

$$\sum_k A(n,k) = f(n) \quad \text{say, for } n = 0, 1, 2, \ldots \qquad (\dagger)$$

(where the range of k may be from $-\infty$ to ∞) is equivalent to showing

$$\sum_k \frac{A(n,k)}{f(n)} = 1 \quad \text{for } n = 0, 1, 2, \ldots \qquad (3)$$

or letting $F(n,k) = A(n,k)/f(n)$ we can write (3) as

$$\sum_k F(n,k) = 1,$$

which is now independent of n. Replacing n by $n+1$, it follows that

$$\sum_k \{F(n+1,k) - F(n,k)\} = 0 \quad \text{for } n \ge 0. \qquad (4)$$

Suppose there exists a "nice" function $G(n,k)$ with the property that

$$F(n+1,k) - F(n,k) = G(n,k+1) - G(n,k), \qquad (5)$$

then the series in (4) results in the telescoping of $G(n,k)$, to wit

$$\sum_{k=-L}^{+M} \{F(n+1,k) - F(n,k)\} = \sum_{k=-L}^{+M} \{G(n,k+1) - G(n,k)\}$$
$$= G(n, M+1) - G(n, -L).$$

Let us further require

$$\lim_{k\to\pm\infty} G(n,k) = 0, \qquad (6)$$

then the identity ($\dagger$) $\sum_k A(n,k) = f(n)$ is proved.

It transpires that for a wide class of identities there are such "nice" functions $G(n,k)$ of the form $R(n,k)F(n,k-1)$ — where $R(n,k)$ is a rational function of n and k — and ($\dagger$) is proved by exhibiting F and R. The F and G obtained thus are referred to as a WZ pair if the conditions (5) and (6) hold. This procedure enables one to use symbolic manipulation packages to carry out the steps by computer, and thus prove identity (†). In particular, Dixon's identity in the generalization of Fjelsted:

$$\sum_k (-1)^k \binom{n+b}{n+k}\binom{b+c}{b+k}\binom{c+n}{c+k} = \frac{(n+b+c)!}{n!\,b!\,c!}$$

is proved firstly by taking

$$R(n,k) = \frac{(c+1-k)(b+1-k)}{2(n+k)(n+b+c+1)}$$

thus $G(n,k) = R(n,k)F(n,k-1)$, and secondly by verifying equations (5) and (6) — for which laborious exercise it would be advisable to avail oneself of Macsyma say.

In general there are few known identities involving sums of products of several binomial coefficients. A spectacular generalization of Dixon's beautiful identity is given by equation 5.31 on p.171 of [5] which must surely be the *non plus ultra* of the species.

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SOME GROUPS OF EXPONENT p

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§1 Introduction.

By the *exponent* of a (finite) group G is meant the least common multiple of the orders of the elements of G. It is a well known elementary exercise that groups of exponent 2 are abelian; and all groups of order p^2, p a prime, are abelian. On the other hand there are examples of non-abelian groups of exponent p $(p > 2)$ and order p^3, or p^4, that go back to Burnside, at least (e.g. [1]). Taking a direct product of a non-abelian group of order p^3, for example, with an elementary abelian p-group of order p^n will, of course, give an example of a non-abelian group of exponent p and of arbitrarily large finite cardinality. However as an example of a non-abelian group of exponent p such a group offers little more than its non-abelian direct factor.

Our interest in examples of such groups was stimulated by questions of W. W. Comfort. We present here a simple construction of an infinite class of non-trivial (i.e. non-abelian and indecomposable) groups of exponent p, $p > 2$.

Observe that to say that a group G is abelian is to say that it is equal to its centre, $z(G)$, so that the larger the centre of G the more abelian, in a sense, is G. Similarly G is abelian if and only if its derived group G' is trivial so that the smaller the derived group, the more abelian is G. It may happen that $z(G)$ is contained in G' in which case G has no hope of being abelian: the larger the centre in G the larger the derived group, the smaller the derived group the smaller the centre. Hopes for commutativity are frustrated just in proportion to their strength. For the purposes of this discussion we encapsulate this idea in the