Problem Page

Editor: Phil Rippon

My first problem this time is a remarkable result about spherical triangles, which was apparently first proved by a computer!

26.1 *Prove that if the area of a spherical triangle is one quarter of the area of the sphere, then the midpoints of its sides form an equilateral spherical triangle with angles of* $90°$.

A discussion of the algebraic verification of theorems in geometry and a BASIC program to prove this result can be found in the article *A new method of automated theorem proving* by Yang Lu ('The mathematical revolution inspired by computing' edited by J. H. Johnson and M. J. Loomes, Oxford University Press, 1991). It might be argued that a computer program cannot tell you why the result holds, in the way that a conventional proof should do.

Next is a problem that I heard recently from my school mathematics teacher, Mr Harold Taylor. It was inspired, he says, by a discussion of the relative sizes of bifurcating blood vessels, given on a television science programme.

26.2 *A pipe from A is split into two smaller pipes at P to supply B and C. Given that the pipe AP costs k times as much per unit length as do PB and PC, determine the position of P so that the total cost is a minimum.*

Now, here is some recent news about one of my older problems. Problem 11.2 asked you to prove that the sequence

$$a_{n+2} = |a_{n+1}| - a_n, \quad n = 0, 1, 2, \ldots, \qquad (1)$$

where a_0, $a_1 \in \mathbf{R}$, is always periodic with period 9. Just before last Christmas, Alan Beardon noticed a connection between this problem and the theory of Hecke groups (certain discrete groups of Möbius transformations). This insight has led to a number of extensions and related results, now being written up by Alan, Shaun Bullett and myself; for example, the sequence

$$a_{n+2} = 2\cos(\pi/p)|a_{n+1}| - a_n, \quad n = 0, 1, 2, \ldots,$$

where $p \in \{2, 3, \ldots\}$ and a_0, $a_1 \in \mathbf{R}$, is always periodic with period p^2. For $p = 3$, we obtain the sequence (1).

Finally, here is a solution to problem 23.2 which appeared in issue 23.

23.2 *Let $s(n)$ denote the number of triples (a, b, c), where a, b, c are positive integers with*

$$a + b + c = n, \quad a \le b \le c \text{ and } a + b > c.$$

Determine a simple formula for $s(n)$.

The motivation behind this counting problem is that each such triple (a, b, c) determines an integer-sided triangle, which is unique up to congruence. We denote the set of such triples by

$$S_n = \{(a, b, c) : a, b, c \in \mathrm{N},\ a + b + c = n,\ a \le b \le c, a + b > c\},$$

and record below the elements of S_n, for $0 \le n \le 10$.

n	S_n	$s(n)$
0		0
1		0
2		0
3	$(1,1,1)$	1
4		0
5	$(1,2,2)$	1
6	$(2,2,2)$	1
7	$(2,2,3), (1,3,3)$	2
8	$(2,3,3)$	1
9	$(3,3,3), (2,3,4), (1,4,4)$	3
10	$(3,3,4), (2,4,4)$	2

On the basis of this table, it is clear that $s(n)$ is somewhat irregular, but it appears that $s(n+3) = s(n)$ if n is odd. Indeed, it is clear that if $(a,b,c) \in S_n$, then $(a+1, b+1, c+1) \in S_{n+3}$ and the reverse implication holds also if n is *odd* (because if $a+b+c$ is odd, then $a+b-c$ is odd, so that

$$\begin{aligned}(a+1)+(b+1) > c+1 &\Longrightarrow a+b > c-1 \\ &\Longrightarrow a+b > c).\end{aligned}$$

Thus

$$s(2m+1) = s(2m+4), \quad m = 0, 1, 2, \ldots, \tag{2}$$

and so the problem reduces to the evaluation of $s(2m)$, $m = 0, 1, 2, \ldots$ To do this, we first prove that

$$s(2m+3) = s(2m) + [\tfrac{1}{2}(m+2)], \tag{3}$$

where $[x]$ denotes the integer part of x. For, if $(a+1, b+1, c+1) \in S_{2m+3}$ but $(a,b,c) \notin S_{2m}$, then

$$a+1+b+1 > c+1 \quad \text{and} \quad a+b \leq c,$$

so that $a+b=c$. Hence

$$2m = a+b+c \Longleftrightarrow a+b = m \Longleftrightarrow (a+1)+(b+1) = m+2.$$

Now, there are $[\frac{1}{2}(m+2)]$ pairs $(a+1, b+1)$ with $a+1 \leq b+1$ and $(a+1)+(b+1) = m+2$, so that (3) follows.

Combining (2) and (3) gives, for $m = 0, 1, 2, \ldots$,

$$s(2m+6) = s(2m) + [\tfrac{1}{2}(m+2)]$$

and hence

$$\begin{aligned}s(2m+12) &= s(2m) + \left[\frac{1}{2}(m+2)\right] + \left[\frac{1}{2}(m+5)\right] \\ &= s(2m) + m + 3.\end{aligned}$$

Applying this recurrence relation repeatedly, we find that if $2m = 12k + 2i$, where $i = 0, 1, 2, 3, 4, 5$ and $k = 0, 1, 2, \ldots$, then

$$\begin{aligned}s(2m) &= s(2i) + (i+3) + (i+9) + \cdots + (i + 6(k-1) + 3) \\ &= s(2i) + 6(k-1)k/2 + k(i+3) \\ &= s(2i) + k(3k+i) \\ &= s(2i) + (m^2 - i^2)/12,\end{aligned}$$

since $k = (m-i)/6$. Thus, in this case,

$$s(2m) - m^2/12 = s(2i) - i^2/12.$$

On examining the table above, we find that, for $i = 0, 1, 2, 3, 4, 5$,

$$s(2i) \text{ is the nearest integer to } i^2/12.$$

Hence, for $m = 0, 1, 2, \ldots$,

$$s(2m) \text{ is the nearest integer to } m^2/12,$$

so that, by (2),

$$s(2m+1) \text{ is the nearest integer to } (m+2)^2/12.$$

To get some feeling for this formula, it is a nice exercise to find the first value of n for which $s(n) > n$.

Phil Rippon,
Faculty of Mathematics,
The Open University,
Milton Keynes MK7 6AA,
UK.