PERFECT COMPACT T_1 SPACES

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Abstract: In a previous note in this bulletin, M. Ó Searcóid [6] proved several interesting results on perfect sets. In this article we prove some results on the existence of largish perfect sets (§1), use an Erdös-Rado partition relation to bound cardinalities (§2) and complete the cardinality picture in the final section.

§1. Perfect sets in perfect spaces. The existence of a perfect set in a topological space is a far from certain thing. Even compact Hausdorff spaces do not in general have perfect sets: the ordinal spaces in the order topology constitute the classical counterexample. For this reason one seeks representability conditions which imply the existence of perfect subsets. The condition studied in this section is the following: X is a *perfect* space if and only if every closed set is a G_δ. Examples are the reals, any metric space, any discrete space...; indeed for any topology T on X there is a smallest topology $T' \supset T$ in which X is a perfect space. The main result of §1 says that if X is an uncountable perfect Lindeloef T_1 space, then X contains a perfect subset P of cardinality $|X|$, and $X - P$ is countable. In other words, there is a Cantor-Bendixson theorem for perfect Lindeloef T_1 spaces too.

Definition. For $A \subset X$, $A' := \{x \in A : \text{for all } N \in N_x |A \cap N| > 1\}$ where N_x is the family of open neighbourhoods of x. For each ordinal α define $X^0 := X$, $X^{\alpha+1} := (X^\alpha)'$ and $X^\alpha := \cap_{\beta<\alpha} X^\beta$ if α is a limit ordinal. We use ω for the (cardinality of) the set of natural numbers.

Lemma 1. *For all $\alpha < \omega_1$, (the first uncountable cardinal) if X is uncountable perfect Lindeloef T_1, then (1) $X^\alpha - X^{\alpha+1}$ is countable; (2) $|X^\alpha| = |X|$; (3) $X - X^\alpha$ is countable.*

Proof. X is perfect Lindeldoef T_1, so X^α is also (since X^α is closed in X). Prove (1), (2) and (3) by induction on α. For $\alpha = 0 : X - X'$ is a discrete subset of X, X' is closed in X, so $X' = \cap_{n\in\omega} G_n$, G_n open, and so $X - X' = \cup_{n\in\omega} F_n$, F_n closed. If $|X - X'| > \omega$, then for some $n |F_n| > \omega$; F_n is Lindeloef and discrete—a contradiction. Thus (1), (2) and (3) hold. For $\alpha = \beta + 1$: from (1), (2) and (3) of the inductive hypothesis for β, $|X^\alpha| = |X^\beta| = |X|$, so X^α inherits the uncountability condition too, and (as for $\alpha = 0$) $|X^\alpha - X^{\alpha+1}| \leq \omega$, $|X - X^\alpha| = |\cup_{\gamma\leq\beta} X^\gamma - X^{\gamma+1}| \leq |\beta| \cdot \omega \leq \omega$. For α a limit ordinal: $X^\alpha := \cap_{\beta<\alpha} X^\beta$ so $|X - X^\alpha| = |\cup_{\beta<\alpha} X^\beta - X^{\beta+1}| \leq |\alpha| \cdot \omega = \omega$; hence $|X^\alpha| = |X|$ and again X^α inherits all the conditions on X and so $|X^\alpha - X^{\alpha+1}| \leq \omega$.

Lemma 2. *There exists $\alpha < \omega_1$ such that $X^\alpha = X^{\alpha+1}$.*

Proof. Suppose not. Then for each α and $x \in X^\alpha - X^{\alpha+1}$, there exists $V(x)$ an open neighbourhood of x with $V(x) \cap X^{\alpha+1} = \phi \cdot X^{\omega_1} = \cap_{\alpha<\omega} X^\alpha$ is closed so $X - X^{\omega_1} = \cup_{\alpha<\omega_1} X^\alpha - X^{\alpha+1} = \cup_{n\in\omega} F_n$, F_n closed.

By lemma 1, $X^\alpha - X^{\alpha+1}$ can be enumerated as $\langle x(n,\alpha) : n \in \omega\rangle$. Thus $\cup_{n\in\omega} F_n = \{x(n,\alpha) : \alpha < \omega_1, n < \omega\}$, so for some m, some $B \subseteq \omega_1$, B cofinal (unbounded) in ω_1, and C_α, $\alpha \in B$, $\phi \neq C_\alpha \subset \omega$ one has:

$$F_m = \{x(n,\alpha) : \alpha \in B, n \in C_\alpha\}.$$

Now $\{V(x(n,\alpha)) \cap F_m : \alpha \in B, n \in C_\alpha\}$ is an open cover of F_m (closed hence Lindeloef), so for some countable $A \subset B$, and $D_\alpha \subset C_\alpha (\alpha \in A)$

$$F_m \subseteq \bigcup \{V(x(n,\alpha)) : \alpha \in A, n \in D_\alpha\} \qquad (*)$$

But $\sup A < \omega_1$ since A is a countable set of countable ordinals. So one can find $\beta \in B - (\sup A + 1)$. Consider $x(r,\beta)$ for any $r \in C_\beta : x(r,\beta) \in \cup\{V(x(n,\alpha)) : \alpha \in A, n \in D_\alpha\}$ since $x(r\beta) \in X^{\alpha+1}$ and $V(x(n,\alpha)) \cap X^{\alpha+1} = \phi$ for all $\alpha \in A$ and $n \in D_\alpha$. Of course $x(r,\beta) \in F_m$—contradicting (*).

Therefore there exists $\alpha < \omega_1$ with $X^\alpha = X^{\alpha+1}$

Theorem 3. *X contains a perfect subset P of cardinality $|X|$, and $X - P$ is countable.*

Proof. Choose the first $\alpha < \omega_1$ such that $X^\alpha = X^{\alpha+1}$. Then $P := X^\alpha$ is required.

§2 Partition relations and the power of perfect Lindeloef T_1 spaces.

Question: if from a palette of μ colours one assigns a colour to each n element subset of a set X, can one be sure of finding a large subset $H \subset X$ which is monochromatic: every n element subset of H receives the same colour? It depends. The study of this kind of problem by F. P. Ramsey and later by P. Erdös and co-workers initiated the partition calculus [2], whose many applications include a proof of a famous theorem of Arhangel'skiĭ that every first countable compact (or even Lindeloef) Hausdorff space has power at most continuum (also in [2]).

Some notation: $[X]^n := \{A \subset X : |A| = n\}$; for cardinals $\kappa, \lambda, \mu : \kappa \rightarrow (\lambda)^n_\mu$ read: "κ arrows λ super n sub μ") abbreviates the statement: for every set X of power κ, for every function $f : [X]^n \rightarrow \mu$, there exist $H \subset X$ and $\alpha < \mu$ such that (i) $|H| = \lambda$ (ii) for every $A \in [H]^n$, $f(A) = \alpha$. Intuitively speaking, the partition relation $\kappa \rightarrow (\lambda)^n_\mu$ holds if for every colouring f of $[X]^n$ by μ colours, there is a monochromatic (homogeneous) $H \subset X$ of power λ.

For orientation, here are some partition relations which are theorems of ZFC (Zermelo-Fraenkel set theory with the axiom of choice). (κ^+ is the next cardinal after κ.)

Theorem. *([2], [4], [1])*

(1) $\omega \rightarrow (\omega)^n_k$ *$n, k \in \omega$ (Ramsey)*

(2) $(\exp_n(\lambda))^+ \rightarrow (\lambda^+)^{n+1}_\lambda$ *(Erdös-Rado)*
where $\exp_0(\lambda : \lambda$, $\exp_{n+1}(\lambda) := \exp_n(2^\lambda)$;

(3) $2^\lambda \not\rightarrow (\lambda^+)^2_2$ *(Sierpiński)*
where $\not\rightarrow$ *means that the relation is false.*

We'll need only the special case $\lambda = \omega$, $n = 1$ of [2]:

$$(2^\omega)^+ \rightarrow (\omega_1)^2_\omega. \qquad (**)$$

Theorem 4. *[7]: If X is perfect Lindeloef T_1, then X has power at most 2^ω.*

The proof of Theorem 4 illustrates very well how partition relations bound cardinality:

Lemma 5. *If X is a topological space in which every singleton subset is a G_δ and if X has no uncountable discrete subsets, then X has power at most 2^ω.*

Proof: For $x \in X$, $\{x\} = \bigcap_{n\in\omega} G(n,x)$, $G(n,x)$ open. Set $U(n,x) := \bigcap_{m\leq n} G(m,x)$ so that $U(n,x)$ is open, $U(n_1,x) \supseteq U(n_2,x)$ for $n_1 \leq n_2$ and $\bigcap_{n\in\omega} U(n,x) = \{x\}$. $x \neq y \in X$ implies that there exists $k \in \omega$ such that:

$$(*)_k \begin{cases} x \in U(k,x) & y \notin U(k,x) \\ y \in U(k,y) & x \notin U(k,y) \end{cases}$$

Define $f : [X]^2 \rightarrow \omega$ by $f(\{x,y\}) :=$ the least k such that $(*)_k$ holds.

Suppose now that $|X| \geq (2^\omega)^+$. Then by (**) there exist $H \subset X$ and $k_0 \in \omega$ such that

(i) $$|H| = \omega_1$$

and

(ii) for $$x \neq y \in H f(\{x,y\}) = k_0.$$

H is discrete, for if $y \neq x \in H$, then by (ii) $y \in U(k_0,x)$ so that $H \cap U(k_0,x) = \{x\}$.

To finish the proof of Theorem 4, one employs the simple

Lemma 6. *If X is perfect Lindeloef T_1, then every discrete subset of X is countable.*

Proof. Let $Y \subset X$ be discrete. Put $F := \{x \in X :$ for all $N \in N_x |Y \cap N| > 1\}$. It's easy to check that

(i) F is closed in X and

(ii) $Y \cap F = \emptyset$, $Y \subset X - F$ and Y is closed in $X - F$. From (i) and (ii) Y is an F_σ in X, say $Y = \cup_{n\in\omega} F_n$, F_n closed in X. Since Y is discrete and X is Lindeloef, F_n is discrete and Lindeloef, hence countable. So $|Y| \leq \sum_{n\in\omega} |F_n| \leq \omega \cdot \omega = \omega$.

Proof of Theorem 4. Every singleton subset is closed in a T_1 space; apply lemmas 5 and 6.

Remarks:

(1) Lemma 6 and Theorem 4 are from [7]; Lemma 5 comes from [3].

(2) Similarly it is easy to show: if X is a perfect κ^+-compact T_1 space, then X has power at most 2^κ.

(3) Recall that X has the Souslin property (the countable chain condition) if and only if there is no uncountable family of pairwise disjoint non-empty open subsets of X. Using (**) one can prove that if X is a first countable Hausdorff space with the Souslin property, then X has power at most 2^ω.

§3Uncountable perfect compact T_1 spaces have power 2^ω. Theorem 4 says that uncountable perfect compact T_1 spaces have power at most 2^ω. In fact any such space has power exactly 2^ω.

Lemma 7. *If $A \subset X$ is a closed uncountable set, then it is possible to find disjoint closed sets B, C, $B \subset A$ with $A \cap B$, $A \cap C$ both uncountable.*

Proof. Choose $a \in A$ such that $A \cap G$ is uncountable whenever Ga is open. (a exists, since otherwise for all $a \in A$ there exists $G(a)$ open, $a \in G(a)$ and $G(a) \cap A$ is countable; A is compact and so $A \subset (G(a_1) \cup G(a_2) \cup \cdots \cup G(a_n)) \cap A$ giving $|A| \leq \omega$ contradiction.)

$\{a\} = \cap_{n\in\omega} G_n$, G_n open since X is perfect T_1.

$$\left|\{a\} \cup \bigcup_{n\in\omega} (A - G_n)\right| = |A|,$$

so for some n, $A - G_n$ is uncountable. $a \in G_n$ implies that $A \cap G_n$ is uncountable; also $G_n = \bigcup_{m\in\omega} F_m$, F_m closed, so for some m, $a \cap F_m$ is uncountable. Now $B := A \cap (X - G_n)$ and $C := A \cap F_m$ are as required.

Corollary 8. *If $P \subset X$ is perfect uncountable, then there exist P_1, P_2 disjoint uncountable perfect subsets of P.*

Proof Split P to find B, C as in Lemma 7; by Theorem 3 there are $P_1 \subset B$ and $P_2 \subset C$ perfect, $|P_1| = |B|$, $|P_2| = |C|$ and $P_1 \cap P_2 = \emptyset$.

Theorem 9. *Suppose that X is a perfect compact T_1 space. If X is uncountable then X has power 2^ω.*

Proof By Theorem 3, X contains a perfect subset P, $|P| = |X|$. It's enough to show $|P| = 2^\omega$.

Define by induction on ${}^{<\omega}2$ (finite sequences of 0's and 1's) a family of sets P_s for $s \in {}^{<\omega}2$ as follows:

$P_{<>} := P$ ($<>$ is the empty sequence in ${}^{<\omega}2$); if $s \in {}^{<\omega}2$ and P_s is defined so that P_s is uncountable and perfect, choose $P_{s\bar{0}}$, $P_{s\bar{1}}$, disjoint uncountable perfect subsets of P_s (by Corollary 8).

Now define for $f \in {}^\omega 2$ (functions form ω into $\{0,1\}$)

$$P_f := \bigcap_{n\in\omega} P_{f|n}$$

where $f|n$ is the restriction of f to n giving the finite sequences $\langle f(0), f(1), \ldots, f(n-1)\rangle$ in ${}^{<\omega}2$.

For $m \in \omega$, $\bigcap_{n\leq m} P_{f|n} \neq \emptyset$, so by compactness, $P_f \neq \emptyset$.

Thus $\{P_f : f \in {}^\omega 2\}$ si a family of pairwise disjoint non-empty subsets of P. So $2^\omega \leq |P| = |X| \leq 2^\omega$ (By Theorem 4).

Remarks

(1) Some representability condition is necessary, as evidenced by the space $\omega_1 + 1$ with $2^\omega > \omega_1$; similarly a discrete space of power ω_1 with $2^\omega > \omega_1$ indicates the necessity of some degree of compactness.

(2) Theorem 9 resembles the classical theorem that a first countable compact Hausdorff space is either countable or has power 2^ω.

(3) It turns out that Theorem 9 is true under the weaker assumption: if X is compact T_1 and every point of X is a G_δ, then

either $|X| \leq \omega$ or $|X| = 2^{\omega}$. Some of the proof can be found in [5].

References

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Book Review

DIFFERENTIAL EQUATIONS: A DYNAMICAL SYSTEMS APPROACH, PART I

Texts in Applied Mathematics 5

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Reviewed by Donal O'Regan

The book of Hubbard and West provides roughly about one third of a year's undergraduate course in ordinary differential equations for senior undergraduate mathematics students. The authors give a very nice up to date treatment of first order (one dimensional) ordinary differential equations in normal form, namely $x' = f(t,x)$; their own software package MacMath is used and referred to throughout the text to compliment the material.

The book consists of five chapters. Chapter 1 is devoted to qualitative description of solutions; Hubbard and West begin with a discussion of such standard topics as direction fields and computer graphics. However the major part of the chapter is devoted to the introduction of the terms fences, funnels and antifunnels. The authors motivate and illustrate very convincingly how these concepts can be used to examine the behaviour of solutions. Chapter 2 discusses standard methods for solving differential equations analytically; here Hubbard and West provide some lovely insights into some very well known problems. Numerical solutions of differential equations are examined in chapter 3. Here the standard one step methods are discussed and again a very nice treatment is given. Chapter 4 is devoted to the study of existence and uniqueness of solutions. In addition the error bounds stated