EXACTNESS IN ELEMENTARY DIFFERENTIAL EQUATIONS

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ABSTRACT *A simple pattern from linear algebra is present in linear differential equations, recurrence relations and matrix theory.*

If $T : X \to Y$ and $S : Y \to Z$ are abelian group homomorphisms we shall call the pair (S,T) *(left,right) one-one* ([3] Ch 10) if there is inclusion

0.1 $$S^{-1}(0) \subseteq T(X),$$

and *exact* if in addition

0.2 $$ST = 0.$$

Sufficient for (0.1) is that there are homomorphisms $T' : Y \to X$ and $S' : Z \to Y$ for which

0.3 $$S'S + TT' = I;$$

when $T : X \to Y$ and $S : Y \to Z$ are continuous homomorphisms of topological groups, or linear between vector spaces, we shall require that S' and T' are also continuous, or linear. If in particular

0.4 $$T = \begin{pmatrix} A \\ B \end{pmatrix} : X \to \begin{pmatrix} X \\ X \end{pmatrix} \ , \ S = (-B \quad A) : \begin{pmatrix} X \\ X \end{pmatrix} \to X$$

then condition (0.1) takes the form

0.5 $$Ax = By \Longrightarrow x = Bz \ , \ y = Az \ ,$$

while condition (0.2) reduces to commutivity

0.6 $$BA = AB.$$

From (0.5) it follows in particular that

0.7 $$B^{-1}(0) \subseteq A\ B^{-1}(0) \ ,$$

and hence also that

0.8 $$(BA)^{-1}(0) \subseteq B^{-1}(0) + A^{-1}(0) \ .$$

Already this captures a familiar observation [2],[4] about linear equations with constant coefficients : with $D : X \to X$ the operation of differentiation on the space $X = C^\infty(\Omega)$ of infinitely differentiable real, or complex, functions on an open interval $\Omega \subseteq \mathbf{R}$, we have

Theorem 1. *If $p = qr$ is the product of polynomials q and r without nontrivial common factors then*

1.1 $$p(D)^{-1}(0) = q(D)^{-1}(0) + r(D)^{-1}(0).$$

Proof. The Euclidean algorithm gives polynomials q',r' for which

1.2 $$q'q + r'r = \mathrm{hcf}(q,r) = 1 \ ;$$

since everything commutes we can now argue, with $A = q(D)$, $B = r(D)$, $A' = q'(D)$ and $B' = r'(D)$,

$$By = 0 \Longrightarrow y = AA'y \text{ with } BA'y = A'By = 0$$

and hence

$$BAx = 0 \Longrightarrow x = (I - A'A)x + A'Ax \in A^{-1}(0) + B^{-1}(0).$$

This is inclusion one way in (1.1), and the reverse is clear •

Something very similar to Theorem 1 is relevant to elementary matrix theory: if $p(A) = 0$ and $p = qr$ with $\mathrm{hcf}(q, r) = 1$ then [4]

1.3 $$q(A)^{-1}(0) = r(A)(X).$$

With $q = (z - \lambda)^k$ and $r(\lambda) \neq 0$ this shows that the *eigenvectors* of A lie in the column spaces of related polynomials $r(A)$. The conditions (0.2) and (0.3) say something about the solution of equations with coefficients S or T:

Theorem 2. *If*

2.1 $$S'S + TT' = I \text{ and } ST = 0$$

and

2.2 $$T'T + WW' = I \text{ and } TW = 0$$

then

2.3 $$Tx = b \Longrightarrow x = T'b + WW'b \in T'b + T^{-1}(0)$$

and

2.4 $$x = T'b \Longrightarrow Tx = (I - S'S)b.$$

Proof. Clear•

The operations of differentiation and integration fit together in the pattern of (0.2) and (0.3): if $0 \in \Omega$ define operators D, D' and J on the space $X = C^\infty(\Omega)$ by setting

2.5 $$(Dx)(t) = \frac{dx(t)}{dt} \; ; \; (D'x)(t) = \int_{s=0}^{t} x(s)ds \; ; \; (Jx)(t) = x(0) \; ;$$

then evidently

2.6 $$DD' = I = D'D + J \text{ with } DJ = 0 = JD' .$$

If $f \in X$ is arbitrary define multiplications L_f and E_f by setting

2.7 $$(L_f x)(t) = f(t)x(t) \; ; (E_f x)(t) = e^{f(t)}x(t) \; ;$$

then also

2.8 $$L_g L_f = L_{gf} = L_f L_g \; ; \; JL_f = L_{Jf} J \; ; \; JE_f = E_{Jf} J$$

and

2.9 $$\begin{aligned} E_{-f}E_f &= I = E_f E_{-f} \; ; \\ DL_f &= L_f D + L_{Df} \; ; \\ DE_f &= E_f(D + L_{Df}). \end{aligned}$$

It is clear that we can take $T = D$ in Theorem 2 to obtain the familiar form of the solution of the equation $Dx = f$; the same extends to the *first order linear* equation:

Theorem 3. *If* $T = D + L_{Df}$ *then*

3.1 $$TT' = I = T'T + WW' \text{ with } TW = 0$$

with

3.2 $$T' = E_{-f}D'E_f \; ; \; W = E_{-f}J \; ; \; W' = E_f \; .$$

Proof. Again clear•

For second and higher order linear equations there is the technique of *variation of parameters*: we claim that this also can be described by Theorem 2 . The ideas are clear from equations of order two:

Theorem 4. *If* $T = D^2 + L_p D + L_q$ *is second order linear with*

4.1 $$T^{-1}(0) = D^{-1}(0)f + D^{-1}(0)g$$

then

4.2 $$TT' = I = T'T + WW' \text{ with } TW = 0,$$

where

$$T' = \begin{pmatrix} L_f & L_g \end{pmatrix} \begin{pmatrix} D'L_h & 0 \\ 0 & D'L_h \end{pmatrix} \begin{pmatrix} -L_g \\ L_f \end{pmatrix};$$

4.3 $$W = \begin{pmatrix} L_f J & L_g J \end{pmatrix};$$

$$W' = L_h \begin{pmatrix} L_{Dg} & -L_g \\ -L_{Df} & L_f \end{pmatrix} \begin{pmatrix} I \\ D \end{pmatrix}$$

with

4.4 $$1/h = \det H \text{ with } H = \begin{pmatrix} f & g \\ Df & Dg \end{pmatrix}.$$

Proof. We follow the usual "variation of parameters" argument, noting that the *Wronskian matrix* H must be invertible, and in effect make the familar substitution:

4.5 $$L_H \begin{pmatrix} U \\ V \end{pmatrix} = \begin{pmatrix} I \\ D \end{pmatrix}, \text{ giving } L_H \begin{pmatrix} D & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix} = \begin{pmatrix} 0 \\ T \end{pmatrix}.$$

It follows

$$\begin{pmatrix} D'DU \\ D'DV \end{pmatrix} = \begin{pmatrix} D' & 0 \\ 0 & D' \end{pmatrix} L_H^{-1} \begin{pmatrix} 0 \\ T \end{pmatrix} = \begin{pmatrix} D' & 0 \\ 0 & D' \end{pmatrix} \begin{pmatrix} L_\phi \\ L_\psi \end{pmatrix} T$$

giving

$$\begin{pmatrix} U \\ V \end{pmatrix} = \begin{pmatrix} D' & 0 \\ 0 & D' \end{pmatrix} \begin{pmatrix} L_\phi \\ L_\psi \end{pmatrix} T + \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix}$$

and hence

$$I = \begin{pmatrix} L_f & L_g \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix}$$

4.6 $$= \begin{pmatrix} L_f & L_g \end{pmatrix} \begin{pmatrix} D' & 0 \\ 0 & D' \end{pmatrix} \begin{pmatrix} L_\phi \\ L_\psi \end{pmatrix} T +$$

$$\begin{pmatrix} L_f & L_g \end{pmatrix} \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix}.$$

This gives $T'T + WW' = I$; it is left to the reader to check that $TW = 0$ and $TT' = I$•

Of course the coefficients p and q in the operator T are determined by the complementary functions f and g:

4.7 $$L_H \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} -D^2 & 0 \\ 0 & -D^2 \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix}.$$

We can make a similar analysis of *recurrence relations*. Define operators U and V on the space of all real sequences by setting

4.8 $$(Vx)_n = x_{n+1},\ (Ux)_0 = 0 \text{ and } (Ux)_{n+1} = x_n\ :$$

these are the backward and forward *shifts*, and satisfy

4.9 $$VU = I = UV + K \text{ with } KU = 0 = VK\ ,$$

where $(Kx)_0 = x_0$ and $(Kx)_{n+1} = 0$. If we introduce operators L_p, E_p and M by setting

4.10 $$\begin{aligned} (L_p x)_n &= p_n x_n, \\ (E_p x)_0 &= x_0, \\ (E_p x)_{n+1} &= p_0 p_1 \ldots p_n x_{n+1}, \\ (Mx)_n &= x_0 x_1 \ldots x_n \end{aligned}$$

then

4.11 $$L_p E_p = L_{Mp},\ VL_p = L_{Vp}V,\ UL_p = L_{Up}U,\ VE_p = L_p E_p V$$

and hence

4.12 $$(V - L_p)E_p = L_p E_p (V - I).$$

The *first order linear* recurrence relation is the equation $(V - L_p)x = q$:

Theorem 5. *If* $T = V - L_p$ *then*

5.1 $$TT' = I = T'T + E_p J \text{ with } TE_p J = 0,$$

where $(Jx)_n = x_0$ *and*

5.2 $$T'(x_0, x_1, x_2, \ldots) = (0, x_0, p_1 x_0 + x_1, p_2 p_1 x_0 + p_2 x_1 + x_2, \ldots).$$

Proof. Clear•

We can see analogy with differential equations if we write

5.3 $D = V - I,\ D' = SU = S - I$ where $(Sx)_n = x_0 + x_1 + \ldots + x_n$,

giving

5.4 $$DD' = I = D'D + J \text{ with } DJ = 0 = JD' .$$

If p_n never vanishes we have

5.5 $$T' = E_p S E_p^{-1} U = WU$$

where

$$W(x_0, x_1, x_2, x_3, \ldots) =$$

5.6 $$(x_0, p_0x_0 + x_1, p_1p_0x_0 + p_1x_1 + x_2,$$
$$p_2p_1p_0x_0 + p_2p_1x_1 + p_2x_2 + x_3, \ldots)$$

Note also

5.7 $$VJ = J,\ JU = 0,\ SU = US,\ (V - I)S = V.$$

In higher dimensions suppose $\Omega \subseteq \mathbf{R}^3$ is open connected and "starlike" with respect to $0 \in \mathbf{R}^3$, and look at ([1] Ch 5 §3) *differential forms*

5.8 $$w = w_0 + \sum_{r=1}^{3} \sum_{|j|=r} w_j dx_j \ (w_j \in X = C^\infty(\Omega))$$

and *differentiation*

5.9
$$D : w_0 + \sum_{r=1}^{3} \sum_{|j|=r} w_j dx_j \to \sum_{i=1}^{3} D_i w_0 dx_i + \sum_{i=1}^{3} \sum_{r=1}^{2} \sum_{|j|=r} D_i w_j dx_{ij} \ ;$$

here $(D_1f)(a) = \lim_{t\to 0}(f(a_1 + t, a_2, a_3) - f(a_1, a_2, a_3))/t$ etc. (partial differentiation) and $dx_{ij} = dx_i \wedge dx_j$ (exterior multiplication). Diagrammatically:

5.10
$$X \xleftarrow{(D_3\ \ -D_2\ \ D_1)} \begin{pmatrix}X\\X\\X\end{pmatrix} \xleftarrow{\begin{pmatrix}-D_2 & D_1 & 0\\ -D_3 & 0 & D_1\\ 0 & -D_3 & D_2\end{pmatrix}} \begin{pmatrix}X\\X\\X\end{pmatrix} \xleftarrow{\begin{pmatrix}D_1\\D_2\\D_3\end{pmatrix}} X$$

Since the D_j commute, this sequence forms a "chain". The *homotopy* H is derived from *multiplications* L_j and *moments* S_j (weighted radial averages), given by

5.11 $$(L_jf)(a) = a_jf(a), \quad (S_jf)(a) = \int_{t=0}^{1} t^{j-1} f(ta)dt \ ;$$

specifically

$$Hw_0 = 0 \ ;$$

5.11
$$H(w_1dx_1 + w_2dx_2 + w_3dx_3) = (S_1w_1)x_1 + (S_1w_2)x_2 + (S_1w_3)x_3 \ ;$$

$$\begin{aligned} H(w_{12}dx_{12} + w_{13}dx_{13} + w_{23}dx_{23}) = &\\ (S_2w_{12})(x_1dx_2 - x_2dx_1)&\\ + (S_2w_{13})(x_1dx_3 - x_3dx_1)&\\ + (S_2w_{23})(x_2dx_3 - x_3dx_2)\,;&\\ H(w_{123}dx_{123}) = (S_3w_{123})(x_1dx_{23} - x_2dx_{13} + x_3dx_{12})\,.& \end{aligned}$$

Diagramatically:

$$\begin{pmatrix}X\\X\\X\end{pmatrix} \xleftarrow{\begin{pmatrix}-L_2S_2 & -L_3S_2 & 0\\ L_1S_2 & 0 & -L_3S_2\\ 0 & L_1S_2 & L_2S_2\end{pmatrix}} \begin{pmatrix}X\\X\\X\end{pmatrix} \xleftarrow{\begin{pmatrix}L_3S_3\\-L_2S_3\\L_1S_3\end{pmatrix}} X$$

and

5.12 $$X \xleftarrow{(L_1S_1\ \ L_2S_1\ \ L_3S_1)} \begin{pmatrix}X\\X\\X\end{pmatrix}$$

Theorem 6

6.1 $$HD + DH = I - J$$

where

6.2 $$J(w_0 + \sum_{r=1}^{3} \sum_{|j|=r} w_j dx_j) = w_0(0)\underline{1}.$$

Proof. Note the commutation rules

$$D_i D_j - D_j D_i = S_i S_j - S_j S_i = L_i L_j - L_j L_i = 0 \ ;$$

6.3 $$\sum_{i=1}^{3} L_i S_1 D_i = I - J \ ; \ \sum_{i=1}^{3} L_i S_{k+1} D_i = I - kS_k \text{ if } k \geq 1 \ ;$$

$$D_i L_j = \delta_{ij} I + L_j D_i \ ; \ D_i S_k = S_{k+1} D_i \ ; \ L_j S_{k+1} = S_k L_j \ .$$

References

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