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Note on the Diophantine Equation

$$x^x y^y = z^z$$

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In a letter to the Editor of the Irish Times, Dr. Des McHale issued the challenge of finding any solution (x, y, z), with none of x, y, $z = 1$, of the Diophantine equation

$$x^x y^y = z^z.$$

This had appeared as a problem in the first Irish Universities Mathematical Olympiad and apparently none of the contestants found a non-trivial solution. The purpose of this note is to indicate a method for generating solutions to this equation.

Lemma: Suppose X, Y, Z, φ are natural numbers such that

(i) $X + Y - Z = 1$ and

(ii) $\varphi \geq 2$ and

(iii) $\varphi = Z^Z/(X^X Y^Y)$;

then $x = \varphi X, y = \varphi Y, = \varphi Z$ have the property that

$$x^x y^y = z^z.$$

Proof: Consider $x^x y^y$: this equals

$$(\varphi X)^{\varphi X}(\varphi Y)^{\varphi Y} = \varphi^{\varphi(X+Y)}(X^X Y^Y)^{\varphi}.$$

On the other hand z^z equals

$$\varphi^{\varphi Z}(Z^Z)^{\varphi}$$

So $x^x y^y = z^z$ if and only if

$$\begin{aligned}
&\varphi^{\varphi(X+Y)}(X^X Y^Y)^{\varphi} = \varphi^{\varphi Z}(Z^Z)^{\varphi} \\
\Leftrightarrow\ &\varphi^{\varphi(X+Y-Z)}(X^X Y^Y)^{\varphi} = (Z^Z)^{\varphi} \\
\Leftrightarrow\ &\varphi^{\varphi}(X^X Y^Y)^{\varphi} = (Z^Z)^{\varphi} \quad \text{since} \quad X+Y-Z=1 \\
\Leftrightarrow\ &\varphi X^X Y^Y = Z^Z \quad \text{which follows from (iii).}
\end{aligned}$$

Now suppose $X = 2^{2\alpha}$ and $Y = p^{2\beta}$ where p is odd and $\alpha, \beta \geq 1$. Consider $(2^{\alpha} - p^{\beta})^2$. This is

$$2^{2\alpha} + p^{2\beta} - 2^{\alpha+1}p^{\beta} = X + Y - Z$$

say for $Z = 2^{\alpha+1}p^{\beta}$. In this case one has $X + Y - Z = 1$ if and only if

$$(2^{\alpha} - p^{\beta}) = \pm 1. \qquad (*)$$

Subject to this we want to ensure that $Z^Z/X^X Y^Y$ is an integer ≥ 2. Now $\varphi := Z^Z/X^X Y^Y$ in this case can be written as

$$\varphi = \frac{2^{(\alpha+1)[2^{\alpha+1}p^{\beta}]} \cdot p^{\beta(2^{\alpha+1}p^{\beta})}}{2^{\alpha 2^{2\alpha+1}} \cdot p^{2\beta p^{2\beta}}}.$$

The power of 2 in φ equals

$$(\alpha + 1)[2^{\alpha+1} \cdot p^{\beta}] - \alpha 2^{2\alpha+1} \qquad (1)$$

The power of p in φ equals

$$\beta 2^{\alpha+1}p^{\beta} - 2\beta p^{2\beta} \qquad (2)$$

Equation (2) is $\geq 0 \Leftrightarrow 2^{\alpha} - p^{\beta} \geq 0$ (on dividing (2) by $2\beta p^{\beta}$). Therefore in (*) we shall require $2^{\alpha} - p^{\beta} = +1$. Inserting this condition into (1) we get

$$(\alpha + 1)[2^{\alpha+1}(2^{\alpha} - 1)] - \alpha 2^{2\alpha+1} \qquad (3)$$

Dividing by $2^{\alpha+1}$, for (1) to be non–negative we require

$$(\alpha + 1)[2^{\alpha} - 1] - \alpha 2^{\alpha} \geq 0$$

$$\Leftrightarrow 2^{\alpha} - 1 \geq \alpha.$$

However this holds for all $\alpha \geq 1$. Using $2^{\alpha} - p^{\beta} = 1$, (2) becomes $2\beta p^{\beta}$ and (3) simplifies to $2^{\alpha+1}(p^{\beta} - \alpha)$. From this, it is apparent that $\varphi \geq 2$.

Since $2^1 - p = 1$ implies $\varphi = 1$ we shall now assume $\alpha \geq 2, \beta \geq 1$.

Examples:

(i) Choose $\alpha = 2$, then $2^2 - p^{\beta} = 1$ gives $p = 3, \beta = 1$.

Then $X = 2^{2\alpha} = 16, Y = 3^{2\beta} = 9$ and $Z = 2^{\alpha+1}p^{\beta} = 8.3 = 24$. Note that $X + Y - Z = 1$.

Letting $\varphi = Z^Z/X^X Y^Y$, the power of 2 in φ equals $2^{\alpha+1}(p^{\beta} - 2)$ which in this example is 8. The power of p in φ equals $2\beta p^{\beta}$ which equals $2.1.3 = 6$, so

$$\varphi = 2^8 3^6.$$

Hence

$$x = 2^{12} \cdot 3^6, y = 2^8 \cdot 3^8 \quad \text{and} \quad z = 2^{11} \cdot 3^7$$

is a solution of the Diophantine equation

$$x^x y^y = z^z.$$

(ii) Choose any power of 2, say 2^k where $k \geq 2$. Then $p = 2^k - 1$ is always odd and clearly $2^k - p = 1$. So we can take

$$X = 2^{2k}, Y = p^2 \quad \text{and} \quad Z = 2^{k+1}p$$

and compute φ as before. For instance if we take 2^4 then $p = 15$ and we get

$$\begin{array}{lll} X = 2^8, & Y = 225 & \text{and} \quad Z = 480 \\ \varphi = & 2^{352}(15)^{30} & \quad \text{etc.} \end{array}$$

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