ARTICLES

Edge Sums of Hypercubes

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Introduction

A *hypercube* may be defined recursively in terms of the cartesian product of graphs as defined in [1, p.23] and in [2]:

$$Q_n = \begin{cases} K_2 & n = 1 \\ Q_{n-1} \times Q_1 & n \geq 2 \end{cases} \tag{1}$$

The hypercube Q_n of *dimension* n may equivalently be defined as the graph of 2^n nodes such that each node is uniquely labelled with a number expressed as an n-digit binary string and two nodes are adjacent whenever their labels vary in exactly one binary digit. The dimension of the edge uv, where $u = (u_1, \ldots, u_n)$, $v = (v_1, \ldots, v_n)$ is k if $u_k + v_k = 1$ and $u_i = v_i$ for $i \neq k$.

Given such a labeling of Q_n, we form the *network* $N(Q_n)$ by assigning integer weights to the edges of Q_n as follows. For two adjacent nodes i and j, let w_{ij} be the weight on edge $ij \in E(Q_n)$, defined by

$$w_{ij} = i + j \tag{2}$$

In words, each edge is assigned as its weight the sum of the two nodes with which it is incident. In Figure 1, the networks formed by labelling Q_1, Q_2 and

Q_3 in this way are shown.

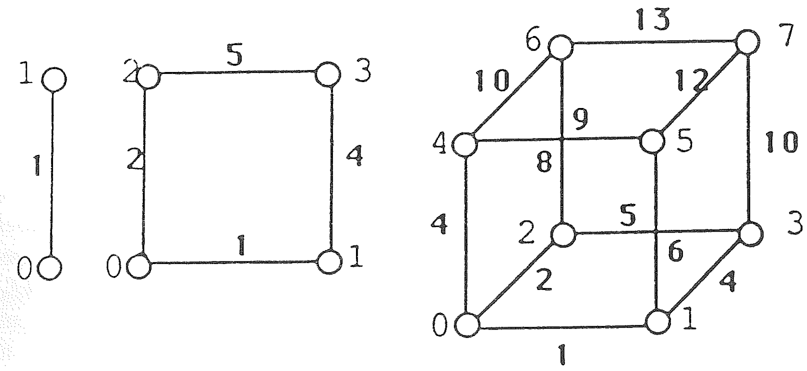

Figure 1.
The labelled hypercubes of dimensions 1,2 and 3 with edge weights.

With each hypercube Q_n we may now associate a set L_n and a multiset M_n of the weights on the edges of $N(Q_n)$. Thus $L_2 = M_2 = \{1,2,4,5\}$ and $L_3 = \{1,2,4,5,6,8,9,10,12,13\}$ while $M_3 = \{1,2,4,4,5,6,8,9,10,10,12,13\}$.

Note that the numbers from 1 to 13 which are not in the set L_3 are 3, 7 and 11. When expressed in binary form, these numbers are 11, 111 and 1011. The fact that they all end in 11 is no coincidence, as we shall see.

Our purpose is to characterize the set

$$L_\infty = \bigcup_{n=1}^{\infty} L_n \tag{3}$$

that is, the set of all integers that are the weights of an edge in some $N(Q_n)$. We will also determine the limiting multiset

$$M_\infty = \lim_{n \to \infty} M_n \tag{4}$$

Of course L_∞ can also be written as the limit of the sets L_n.

Edge Sums

To characterize those numbers which are edge sums for some hypercube, we require the operation of the sum of two multisets of numbers. For multisets $S_1, S_2 \subset \mathbb{Z}^+$, define $S_1 + S_2 = \{w_1 + w_2 : w_i \in S_i\}$. When S_1 is a singleton w_1 we abuse the notation by writing $S_1 + S_2 = w_1 + S_2$.

To derive a recurrence relation for the sets M_n, we appeal to the recursive definition (1) of Q_n. Note that Q_n may be constructed from two copies of Q_{n-1}. One copy of Q_{n-1} has the usual integer values on its nodes, and thus has the multiset M_{n-1} of weights on its edges. The other copy is labelled similarly with node labels all greater by 2^{n-1}. Thus its edge weights are all greater by $2^{n-1} + 2^{n-1} = 2^n$. A pair of nodes, one from each copy of Q_{n-1}, are adjacent whenever their labels differ by 2^{n-1}, as the edges joining them lie in the nth dimension of Q_n. Consequently, M_n may be written as the union of M_{n-1} with the elements of $2^{n-1} + M_{n-1}$, along with the weights on the edges joining the two copies of Q_{n-1}. Thus, since the base case is $M_1 = \{1\}$ as seen in Figure 1, we find that

$$M_n = M_{n-1} \bigcup (2^n + M_{n-1}) \bigcup \{2^{n-1} + 2k : 0 \leq k < 2^{n-2}\} \tag{5}$$

As L_n is the set of the multiset M_n, it follows that

$$L_n = M_{n-1} \bigcup (2^n + L_{n-1}) \bigcup \{2^{n-1} + 2k : 0 \leq k < 2^{n-2}\} \tag{6}$$

for all $n \geq 1$. Let $\mathbb{Z}^+$ be the set of all positive integers and write $\mathbb{Z}_0 = \mathbb{Z}^+ \cup \{0\}$. Using this notation we are now able to state our main result.

Theorem 1 *A positive integer z is the weight of some edge in $N(Q_n)$ for sufficiently large n if and only if $z \not\equiv 3 \pmod 4$. Thus $z \in \mathbb{Z}^+$ is not a hypercube weight just if $z = 4x+3$ for some $x \in \mathbb{Z}_0$, i.e., the binary expression of z ends in 11.*

Proof Let $\alpha 00 \beta$ and $\alpha 10 \beta$ be adjacent nodes in Q_n, where α and β are binary strings. The weight on the edge joining these nodes is then $\alpha 01 \beta 0$, found by base 2 addition. As α and β may both be null strings we see that the resultant string $\alpha 01 \beta 0$ can be any even number. An odd sum may only arise if the summands vary in their least significant digit, that is, between nodes with labels $\alpha 0$ and $\alpha 1$ for some binary string α. Obviously, their sum is $\alpha 01$ and odd numbers of the form $\alpha 11$ cannot be formed in this way. Thus the only admissible odd numbers are those of the form $4x+1$. Combining this with the fact that all even numbers are admissible, the result is established.

As well as characterizing all the integers in L_∞, we may also find the multiplicity of each integer in M_∞.

Corollary 1a *The multiplicity $f(y)$ of the integer y in M_∞ is given by*

$$f(y) = \begin{cases} \lfloor \log_2 y \rfloor & y = 2x \\ 1 & y = 4x+1 \\ 0 & y = 4x+3 \end{cases} \tag{7}$$

Proof We have already shown that $f(4x+3) = 0$. We now consider the other two cases. Without loss of generality let i and j be adjacent nodes in Q_n, with $i < j$. Clearly, $j = i + 2^k$ for some k with $0 \leq k \leq n$. Then the integer y found by adding i to j is

$$y = 2i + 2^k \tag{8}$$

Obviously, when y is odd we require that $k = 0$ and the solution, if any, is unique. We have already seen that when $y = 4x+1$ for some integer $x \geq 0$ such a solution does exist. For even $y = 2m$ we see that $y = i + 2^{k-1}$, which has as many solutions as there are values of 2^{k-1} which are less than m, that is, $2^k < 2m = y$. All such solutions are admissible and thus the multiplicity of $y = 2m$ is given by the number solutions of the diophantine equation (8). The number of such solutions is obviously given by $\lfloor \log_2 y \rfloor$.

Using (6) and (7), we find L_4 and M_4:

$$\begin{aligned} L_4 &= \{1,2,4,5,6,8,9,10,12,13,14,16,17,18,20,21, \\ &\quad 22,24,25,26,28,29\} \\ M_4 &= \{1,2,4,4,5,6,8,8,9,10,10,10,12,12,13,14,16, \\ &\quad 17,18,18,20,20,20,21,22,22,24,25,26,26,28,29\} \end{aligned} \tag{9}$$

Note that (for $n \leq 4$) the sets L_n and multisets M_n are symmetric about the value $2^n - 1$. We now show that this is true in general.

Corollary 1b *A number z is the weight of some edge of Q_n if and only if its reflection about $2^n - 1$, viz., $z + 2(2^n - 1 - z) = 2^{n+1} - 2 - z$, is an edge weight of Q_n.*

Proof Let i and j be adjacent nodes in Q_n expressed as binary strings and let i' and j' be the bitwise complements of i and j so that

$$i + i' = j + j' = 2^n - 1$$

As i and j are adjacent, the complementary nodes i' and j' are adjacent. Thus, for every edge whose weight is z there exists another edge whose weight is $2(2^n - 1) - z$.

References

[1] F. Harary, *Graph Theory*. Addison-Wesley, Reading, 1969.

[2] F. Harary, J.P. Hayes and J-H. Wu, *A survey of the theory of hypercube graphs*, Comput. Math. Appl., to appear.

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Algebraic Techniques Of System Specification

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1 Introduction

The design of complex software systems is a relatively new occupation and is still in its infancy. With the rapid growth in the applications of microprocessor technology more and more areas of life are being affected and in some of this activity there is serious cause for concern. Many manufacturers are using microcomputers to control safety-critical systems. Such systems are usually defined to be systems, the malfunctioning of which could lead directly to injury or death on a small (local) or large (global) scale. Examples of recent products and systems that have caused injury or death through inadequate software design are :

- Chemical processing plants,
- Washing machines,
- Car cruise controls,
- Intensive care systems,
- Industrial robots, etc.

In many of these systems there is a serious problem in the formal specification of the total system and its environmental interaction. Most interactive systems involve three important components :

- the system,
- the environment,
- the user.