where $N(s,n)$ is the number of occurrences of s in the first n digits of π. Because of this, the digits of π are sometimes used in algorithms to generate sequences of random numbers.

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Topological Equivalents of the Axiom of Choice

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Recall that, within the terms of Von Neumann-Bernays-Gödel set theory, one form of the axiom of choice (abbreviated AC) is stated as follows:

> *If* $\{X_i : i \in I\}$ *is a non-empty disjoint family of non-empty sets, then there exists a set* C *such that* $C \cap X_i$ *is a singleton for each* $i \in I$.

The axiom of choice has become virtually indispensable in mathematics since a large number of important results have been obtained from it in almost all branches of the subject without leading to a contradiction. However, although this axiom is consistent with, yet independent of, the other axioms of set theory, its status has long been a source of controversy and not all mathematicians are willing to accept it. Perhaps the principal appeal of the axiom of choice resides in the extensive list of its logical equivalents which exist in apparently disparate areas of mathematics. A fairly comprehensive dossier of these was compiled by the Rubins [4] in 1963.

Most topologists side with the majority of mathematicians, assume the axiom of choice, and do not hesitate to use it whenever necessary. Indeed some would argue that the following proposition (usually known as Tychonoff's theorem) constitutes the single most important result in general topology:

> *The product of a family of non-empty compact topological spaces is compact.*

The point here is that Tychonoff's theorem is logically equivalent to the axiom of choice (see [3]). In this note some other such topological equivalents are introduced.

Classically a topological space (X,τ) is said to be a T_0*-space* (T_1*-space*) if and only if for every pair of distinct points in X there exists a τ-neighbourhood of one which does not contain the other (exist τ-neighbourhoods of each which do not contain the other). Properties like T_0 and T_1, when possessed by a topological space, essentially express a degree of separation enjoyed by the points in the space. A non-empty subset Y of space (X,τ) is said to be *dense* (*codense*) if and only if there exists no non-empty τ-open (τ-closed) subset H of X such that $Y \cap H$ is empty. Let us call Y *thick* if and only if there exists no non-empty τ-open and τ-closed subset H of X such that $Y \cap H$ is empty. Evidently if Y is either dense or codense then it is thick.

Given some topological invariant property P, consider the following statements:

(MP) every topological space (X,τ) has a subspace $(Y,\tau|Y)$ (where $\tau|Y$ is the relativization of τ to Y), with property P, which is maximal (with respect to inclusion);

(DP) every topological space (X,τ) has a subspace $(Y,\tau|Y)$, with property P, which is dense (in (X,τ));

(CP) every topological space (X,τ) has a subspace $(Y,\tau|Y)$, with property P, which is codense (in (X,τ));

(TP) every topological space (X,τ) has a subspace $(Y,\tau|Y)$, with property P, which is thick (in (X,τ)).

It is clear that either of DP or CP implies TP. Schnare [5] showed that MT_0 and MT_1 are each equivalent to AC, and, here, his results are used to confirm that the same is true for DT_0 and CT_0.

Theorem 1 *The following statements are equivalent:*
(i) AC
(ii) DT_0
(iii) CT_0.

Proof (i) implies (ii). Let (X,τ) be any topological space so that, by hypothesis and [5], there exists a maximal T_0 subspace $(Y,\tau|Y)$. Then Y is dense in (X,τ), otherwise there exists a τ-open subset H (of X) which is disjoint from Y and contains a point x, so that, since $\{x\}$ is $\tau|Z$-open, the subspace $(Z,\tau|Z)$ is T_0, thereby contradicting the maximality of Y (where $Z = Y \cup \{x\}$).

(i) implies (iii). Replace the word "dense" by "codense" and the word "open" by "closed" throughout the argument above.

(ii) implies (i). Let $\{X_i : i \in I\}$ be a disjoint family of non-empty sets, let $X = \bigcup\{X_i : i \in I\}$, and consider the partition topology

$$\tau = \{G \subseteq X : G \cap X_i \neq \emptyset \text{ implies } X_i \subseteq G\}$$

If $(Y,\tau|Y)$ is a dense T_0 subspace of (X,τ), then, by its density, Y meets each X_i but, since Y is T_0, in exactly one point.

(iii) implies (i). Replace the word "dense" by "codense" throughout the argument above.

However, DT_1 and CT_1 are false.

Example Let X be the set of real numbers and consider the nested topology $\tau = \{G \subseteq X : G = (a,\infty), a \in X\} \cup \{\emptyset, X\}$ (where (a,∞) denotes the interval $\{x \in X : a < x\}$). It is immediate that any subspace of (X,τ) is nested, that any T_1 subspace is therefore a singleton and bounded, whereas any dense (codense) subspace is unbounded above (below). Observe that any singleton subspace is a maximal T_1 subspace which is neither dense nor codense.

In view of the equivalences obtained by Schnare, if T_α is any hereditary invariant property lying in logical strength between T_0 and T_1 (including those separation axioms discussed in [1] and [2]), it is tempting to conjecture that MT_α is equivalent to AC. So far this remains an open question. Although DT_α implies DT_0 and CT_α implies CT_0, so that, by Theorem 1, each implies AC, the example above seems to suggest (for all known such T_α which are strictly stronger than T_0) that DT_α and CT_α are false. Certainly (mindful that a space is called a *T_{ES}-space* if and only if every singleton subset is either open or closed), in the example, every T_{ES}-subspace is at most a doubleton while, indeed, every doubleton subspace is a maximal T_{ES}-subspace which is neither dense nor codense. That is, for instance, DT_{ES} and CT_{ES} are false.

On the other hand, we have:

Theorem 2. *The following statements are equivalent:*
(i) AC
(ii) TT_a

Proof Since AC implies MT_1, and TT_1 implies TT_α implies TT_0, it only remains to verify that MT_1 implies TT_1 and TT_0 implies AC.

MT_1 implies TT_1: Let (X,τ) be any topological space so that, by hypothesis, there exists a maximal T_1 subspace $(Y,\tau|Y)$. Then Y is thick in (X,τ), otherwise there exists a τ-open and τ-closed subset H (of X) which is disjoint from Y and contains a point x, so that, since $\{x\}$ is $\tau|Z$-open and $\tau|Z$-closed, the subspace $(Z,\tau|Z)$ is T_1 (where $Z = Y \cup \{x\}$), thereby contradicting the maximality of Y.

TT_0 implies AC: Repeat the argument of (ii) implies (i) in Theorem 1, with the word "dense" replaced by "thick".

Remarks It is interesting to contrast and compare MP, DP, CP and TP for a general invariant P. For example, if P is "connected", MP is true (since, as is well known, the maximal connected subspaces are the connected components), DP is false (since, as is well known, the closure of a connected subspace is connected), CP is false (since each connected subspace of a disconnected space, being contained in a component, is therefore disjoint from any other (closed) component) and TP is false (since each connected subspace of a locally connected disconnected space, being contained in a component, is therefore disjoint from any other (open and closed) component).

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HISTORY OF MATHEMATICS

The Culmination Of A Dublin Mathematical Tradition

On The Maxwellian Struggle For A New Mathematical Physics And The Birth Of Relativity

N.D. McMillan

This paper is to celebrate the centenary of the Hertz Exporimatum Crucis that proved the FitzGerald electromagnetic theory of radio transmission.

Fitzgerald And The Electromagnetic Description of Light Propagation.

FitzGerald's chosen field of study in Dublin University for his Fellowship examinations in the period 1871–1877 was MacCullagh's mathematical researches. This study perhaps uniquely prepared him to comprehend the full significance of James Clerk Maxwell's development of an electromagnetic theory of light in 1865, which had until 1879 remained largely ignored, except for a handful of "electricians" from outside of the establishment of science and engineering.

FitzGerald and his uncle George Johnstone Stoney in Dublin, were the first mathematicians from the established universities to see Maxwell's work as the departure point for a programme of mathematical researches that would provide a unifying theory for physics. If successful of course, such a unified theory would have also established Cambridge and Dublin at the unchallenged head of developments in British science. There was at the time a determined ideological challenge to the scientific leadership based on the mathematicians in

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