There is a story (true or false?) that a certain teacher, nameless of course, in a certain school, nameless also, informed the school inspector that he/she advised his/her Intermediate Certificate class to *always* choose B in the multiple choice part of the Mathematics paper as he/she had done a survey of the previous few years and B had come up more often than any other. I believe there will be no multiple choice questions when the new Intermediate Certificate syllabus is examined.

References

[1] N. O'Murchu and C.T. O'Sullivan, *Mathematical Horses for Elementary Physics courses*, I.M.S. Newsletter, 6(1982), 50-54.

[2] *Report on the Basic Mathematical skills test of First Year Students in Cork RTC in 1984*, I.M.S. Newsletter, 14(1985), 33-43.

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NOTES

Wedderburn's Theorem Revisited (Again)

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In a previous note in this Bulletin [3] we proved the following theorem which generalises the theorem of Wedderburn that a finite division ring is a field.

Theorem 1 *Let R be a ring with unity. If more than $|R| - \sqrt{|R|}$ elements of R are invertible, then R is a field.*

The bound $|R| - \sqrt{|R|}$ is the best possible because of the existence of $\mathbf{Z}_{p^2}$, which has exactly $p^2 - p$ invertible elements for any prime p, but yet is not a field.

Another formulation of Wedderburn's theorem is the following: If R is a finite ring with unity and every non-zero element of R is invertible, then R is commutative.

This naturally leads to the following question: If R is a finite ring with unity, can we force the conclusion that R is commutative by assuming that a proper subset of the non-zero elements are invertible? The purpose of this note is to prove the following:

Theorem 2 *Let R be a finite ring with unity. If every non-zero ring commutator $[x, y] = xy - yx$ of R is invertible then R is commutative.*

Proof Let $c = [x, y] \neq 0$. Consider the sequence $c, c^2, c^3, \ldots$ Since R is finite, $c^i = c^j$ for some $j > i \geq 1$. By hypothesis, c is invertible, so $c^{j-i} = 1$ and thus $c^{j-i+1} = c$. R now satisfies the hypothesis of a theorem of Herstein [1], $[a, b]^{n(a,b)} = [a, b]$ for $n(a, b) \geq 1$. If we are prepared to invoke the full power of this theorem, it follows at once that R is commutative. Alternatively, we can use the following more elementary result of Herstein [2]: If R is a finite ring in which every nilpotent element is cenmtral, then R is commutative.

We argue as follows. Let x, y, r be elements of R with $xy \neq 0$. Then $(yx-xy)^n = yx-xy$ implies that $(yx)^n = yx = 0$. Similarly, $(x(ry)-(ry)x)^n = x(ry)-(ry)x$ implies that $xry = 0$. A simple induction argument now shows that all nilpotent elements are central. Thus R is commutative.

Of course, R need not be a field, as the example $(\mathbb{Z}_4, \oplus, \otimes)$ shows.

Finally, we are indebted to Professor T.J. Laffey who has supplied the following ingenious alternative proof of Theorem 1.

Let R be a finite ring with unity 1, let $T = T(R)$ be its group of units and suppose that $T \neq R \setminus \{0\}$. Let $0 \neq x \in R \setminus T$ and let $T_0 = \{ t \in T \mid xt = x \}$. We note that T_0 is a subgroup of T and that $V = \{ xv \mid v \in T \}$ is a subset of $R \setminus (T \cup \{0\})$, with $|V| = |T|/|T_0|$. Let $W = \{ t-1 \mid t \in T_0 \}$. We note that $|W| = |T_0|$ and that $W \subset R \setminus T$, since $t - 1 \in T$ and $xt = x$ implies $x = 0$. Hence $|R| \geq |T| + |V| + 1 = |T| + |T|/|T_0| + 1$ and also $|R| \geq |T| + |T_0|$. Hence we deduce that $|R| - |T| \geq \max(|T_0|, |T|/|T_0| + 1$. So $|R| - |T| \geq \sqrt{|R|} + 1$.

References

[1] I.N. Herstein, *Noncommutative Rings*, Carus Mathematical Monographs, No. 15, Mathematical Association of America, Washington DC, 1968.

[2] I.N. Herstein, *A note on rings with central nilpotent elements*, Proc. Amer. Math. Soc. 5(1954), 620.

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Periodic Functions

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This article arose out of correspondence between the author and Mark Heneghan regarding certain inconsistencies in the treatment of periodic functions in our secondary school texts. A complete and rigorous treatment of this topic requires the introduction of such concepts as convergent sequence, continuity, greatest lower bound, induction and linear independence. We have tried to minimize the impact of these concepts and at the same time to clarify the situation regarding the sum of periodic functions.

Definition 1 A function $f : \mathbb{R} \mapsto \mathbb{R}$ is *periodic* if there exists $a \neq 0$ such that

$$f(x+a) = f(x) \qquad \text{for all } x \in \mathbb{R}. \qquad (1)$$

Any real number a satisfying (1) is called a *period* of f.

Remarks (1) If a is a period of f then so is $-a$, since $f(x) = f(x-a+a) = f(x-a)$.
(2) If a is a period of f and n is an integer then na is also a period of f. This follows from the identity

$$f(x+na) = f(x+(n-1)a+a) = f(x+(n-1)a) \, ,$$

using induction and our first remark.
(3) If a and b are periods of f then $a+b$ is also a period of f, since $f(x+a+b) = f(x+a) = f(x)$.

Example 1 Let f be given by $f(x) = \sin x$. Then f is periodic since $f(x + 2\pi) = f(x)$ for all $x \in \mathbb{R}$.

Example 2 Let f be given by

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational} \end{cases}$$

If a and x are rational and y irrational then $a + x$ is rational and $a + y$ is irrational, and hence $f(x+a) = 0 = f(x)$ and $f(y+a) = 1 = f(y)$. Thus