

follows that  $(X, \tau)$  is not locally connected (at  $x$ ).

#### REFERENCE

1. STEEN, L.A. and SEEBACH, J.A.  
'Counterexamples in Topology', Holt, Rinehart and Winston  
Inc., New York, 1970.

Department of Pure Mathematics,  
Queen's University,  
Belfast BT7 1NN,  
Northern Ireland.

#### R.I.A. PROCEEDINGS

Members of the Irish Mathematical Society benefit from a special discount of one-third of the normal price on subscriptions to Section A of the Proceedings of the Royal Irish Academy.  
Orders may be placed through the I.M.S. Treasurer.

## NONCOMMUTATIVE ANTICOMMUTATIVE RINGS

Stephen Buckley and Desmond MacHale

An associative ring  $R$  is said to be *anticommutative* if  $xy + yx = 0$  for all  $x, y \in R$ . If  $R$  has characteristic 2, then the concepts of commutativity and anticommutativity coincide, but  $\mathbb{Z}_3$ , with the usual addition and trivial multiplication, shows that an anticommutative ring need not have characteristic 2.

If a ring  $R$  satisfies  $x^2 = 0$  for all  $x \in R$  then clearly  $R$  is anticommutative, but not conversely. However, if  $R$  is anticommutative it is easy to verify that  $R$  satisfies each of the following identities.

$$(i) \quad 2x^2 = 0 \quad (ii) \quad (xy - yx)^2 = 0 \quad (iii) \quad x^2y - yx^2 = 0.$$

Frequently, when looking at commutativity theorems for rings, one requires counterexamples to show that certain conditions are not sufficient for commutativity. For example, if  $(xy)^2 = x^2y^2$  for all  $x, y \in R$  and either of the following conditions holds then  $R$  is commutative:

- (a)  $R$  has unity;    (b)  $R$  has no non-zero nilpotent elements.

To show that some such additional condition is necessary, it is enough to produce a non-commutative ring in which  $x^2 = 0$  for all  $x \in R$ . In this note, for finite rings, we pose the question, "what is the order of a smallest noncommutative anticommutative ring?" and show that the answer is 27. Since this number is odd, we see that it is also the answer to the question, "what is the order of a smallest noncommutative ring satisfying the identity  $x^2 = 0$ ?".

First of all we produce a ring of order 27 with the desired properties. Let  $A = (a_{ij})$  be the ring of those  $4 \times 4$  matrices with entries in the field  $Z_3$ , such that  $a_{ij} = 0$  if  $j \leq i$ ,  $a_{23} = 0$ ,  $a_{24} = a_{13}$ , and  $a_{34} = -a_{12}$ . Then it is easily checked that  $R$  is a noncommutative anticommutative ring of order 27. In more abstract terms,  $R$  can be expressed as follows: if  $C_n$  is the cyclic group of order  $n$  and  $\oplus$  denotes the direct sum of groups, then  $(R, +) \cong C_3 \oplus C_3 \oplus C_3 = \langle a \rangle \oplus \langle b \rangle \oplus \langle c \rangle$ , where  $a^2 = b^2 = c^2 = ac = bc = ca = cb = 0$ ,  $ab = -ba = c$  determines the multiplicative operation in  $R$ .

We proceed to show that no ring of order less than 27 can be both noncommutative and anticommutative, so let  $R$  be a ring with these properties. Since every finite ring is the direct sum of rings of prime-power order and since a direct sum of rings is anticommutative if and only if each of its direct summands is anticommutative, we may confine our attention to rings of prime-power order. If  $(R, +)$  is cyclic, then  $R$  is commutative - this eliminates rings of prime order and if  $|R| = p^2$  for some prime  $p$ , we may assume  $(R, +) \cong C_p \oplus C_p$ . Clearly, we may also eliminate rings of characteristic 2. Thus we need only consider the following values of  $|R|$  with corresponding structures for  $(R, +)$ :

- (i)  $|R| = 8$ ,  $(R, +) \cong C_2 \oplus C_4$ ;
- (ii)  $|R| = 9$ ,  $(R, +) \cong C_3 \oplus C_3$ ;
- (iii)  $|R| = 16$ ,  $(R, +) \cong C_2 \oplus C_8, C_2 \oplus C_2 \oplus C_4, C_4 \oplus C_4$ ;
- (iv)  $|R| = 25$ ,  $(R, +) \cong C_5 \oplus C_5$ .

We can eliminate 9 and 25 using the following result.

**LEMMA.** *If  $p$  is an odd prime, then  $C_p \oplus C_p$  cannot be the additive group of a noncommutative anticommutative ring.*

**PROOF.** Let  $R$  be a counterexample and let  $(R, +) = \langle a \rangle \oplus \langle b \rangle$ . Since  $R$  is anticommutative,  $x.x + x.x = 0$  for all  $x \in R$ , so  $x^2 = 0$ , since  $|R|$  is odd. If  $ab = 0$  then  $ab + ba = 0 \Rightarrow$

$ba = 0 = ab$ , so  $R$  is commutative, a contradiction. Suppose that  $ab = ra + sb$  where  $r, s \in Z_p$ . Then  $a^2b = ra^2 + sab$ , so  $sab = 0$ , and so  $s = 0$ . Finally  $ab = ra$ , so  $ab^2 = rab = 0$  which gives  $ab = 0$ , a contradiction.

Next, we suppose that  $(R, +) \cong C_2 \oplus C_4$  or  $C_2 \oplus C_8$  and  $R = \langle a \rangle \oplus \langle b \rangle$ , where  $b$  has order 4 or 8. In either case,  $2ab = (2a)b = 0$ , so  $2ab = ab + ba$ , and  $R$  is commutative. The case  $(R, +) \cong C_2 \oplus C_2 \oplus C_4$  is dismissed in a similar manner. We are left with the possibility that  $(R, +) \cong C_4 \oplus C_4$ . Suppose that  $(R, +) = \langle a \rangle \oplus \langle b \rangle$  where  $4a = 4b = 0$ . Consider first the case where  $a^2 = b^2 = 0$ . Then we get a contradiction, as in the proof of the lemma. Thus we may assume that one generator (a say) satisfies  $a^2 \neq 0$ . Since  $2a^2 = 0$ ,  $a^2 \in (2a, 2b, 2a+2b)$ , the set of elements of order 2 in  $R$ . Suppose first that  $a^2 = 2a$  and let  $ab = ra + sb$ , where  $r, s \in Z_4$ . Then  $2ab = a^2b = a(ab) = ra^2 + sab = 2ra + sab$ . This gives  $(sr)a + s(s-2)b = 0$ . Hence  $s$  is even and if  $s \neq 0$ ,  $r$  is even. This implies that  $ab$  has order 2 and so  $R$  is commutative, a contradiction. Thus  $s = 0$  and  $ab = ra$ ,  $r = \pm 1$ . Then  $ab^2 = (ab)b = rab = r^2a = a$ , so  $2ab^2 = a(2b^2) = 0 = 2a$ , a contradiction.

Finally, we may suppose that  $a^2 = 2b$ , since if  $a^2 = 2a+2b$  we may replace  $b$  in the basis by  $a+b$ . If  $ab = ra + sb$ , we get  $(rs)a + (2r+s^2)b = 0$ . Thus  $s$  is even and if  $s \neq 0$ ,  $r$  is even also, so  $ab$  has order 2, a contradiction. Hence  $s = 0$ , so  $2r + s^2 = 0$ ,  $r$  is even,  $2ab = 0$  and we are finished.

Let  $S$  be the ring of order 32 where  $(S, +) = \langle a \rangle \oplus \langle b \rangle \oplus C_8 \oplus C_4$ , with  $a^2 = 4a$ ,  $b^2 = 2b$ ,  $ab = -ba = 2a$ . Then  $S$  is a noncommutative anticommutative ring of order a power of 2. By our previous analysis,  $S$  is a smallest such 2-ring and in addition,  $S$  is a smallest such ring of even order. Finally, we observe that  $S$  is a smallest ring of the desired type such that  $(S, +)$  is a 2-generator group.

*Department of Mathematics,  
University College Cork.*