Course 424 — Group Representations

Sample Exam

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Attempt 7 questions. (If you attempt more, only the best 7 will be counted.) All questions carry the same number of marks. In this paper representation *means "finite-dimensional representation over* $\mathbb{C}$*".*

1. Define a *group representation*. What is meant by saying that 2 representations α, β are *equivalent*? Find all representations of S_3 of degree 2 (up to equivalence).

   What is meant by saying that a representation α is *simple*? Find all simple representations of D_4 from first principles.

2. What is meant by saying that a representation α is *semisimple*?

   State carefully, and outline the main steps in the proof of, Haar's Theorem on the existence of an invariant measure on a compact group.

   Prove that every representation of a compact group is semisimple.

3. Define the *character* χ_α of a representation α, and show that it is a class function (ie it is constant on conjugacy classes).

   Define the *intertwining number* $I(\alpha, \beta)$ of 2 representations α, β of a group G, and show that if G is compact then

   $$I(\alpha, \beta) = \int_G \overline{\chi_\alpha(g)} \chi_\beta(g)\, dg.$$

   Prove that a representation α is simple if and only if $I(\alpha, \alpha) = 1$.

4. Draw up the character table for S_4.

   Determine also the representation-ring for this group, ie express the product $\alpha\beta$ of each pair of simple representation as a sum of simple representations.

5. Show that the number of simple representations of a finite group G is equal to the number s of conjugacy classes in G.

   Show also that if these representations are $\sigma_1, \dots, \sigma_s$ then

   $$\dim^2 \sigma_1 + \cdots + \dim^2 \sigma_s = |G|.$$

   Determine the dimensions of the simple representations of S_5, stating clearly any results you assume.

6. Determine the conjugacy classes in $\mathbf{SU}(2)$; and prove that this group has just one simple representation of each dimension.

   Find the character of the representation $D(j)$ of dimensions $2j+1$ (where $j = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots$).

   Express each product $D(i)D(j)$ as a sum of simple representations $D(k)$.

7. Define the *exponential* e^X of a square matrix X.

   Determine e^X in each of the following cases:

   $$X = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad X = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

   Show that if X has eigenvalues λ, μ then e^X has eigenvalues e^λ, e^μ.

   Which of the above 5 matrices X are themselves expressible in the form $X = e^Y$ for some real matrix Y? (Justify your answers in all cases.)

8. Define a *linear group*, and a *Lie algebra*; and define the Lie algebra $\mathcal{L}G$ of a linear group G, showing that it is indeed a Lie algebra.

   Define the *dimension* of a linear group; and determine the dimensions of each of the following groups:

   $$\mathbf{O}(n), \mathbf{SO}(n), \mathbf{U}(n), \mathbf{SU}(n), \mathbf{GL}(n, \mathbb{R}), \mathbf{SL}(n, \mathbb{R}), \mathbf{GL}(n, \mathbb{C}), \mathbf{SL}(n, \mathbb{C})?$$

9. Determine the Lie algebras of $\mathbf{SU}(2)$ and $\mathbf{SO}(3)$, and show that they are isomomorphic.

   Show that the 2 groups themselves are *not* isomorphic.

10. Define a *representation* of a Lie algebra $\mathcal{L}$. What is meant by saying that such a representation is (a) *simple*, (b) *semisimple*?

    Determine the Lie algebra of $\mathbf{SL}(2, \mathbb{R})$, and find all the simple representations of this algebra.

    Show that every representation of the group $\mathbf{SL}(2, \mathbb{R})$ is semisimple, stating carefully but without proof any results you need.