Course 424

Group Representations I

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Synge Theatre Monday, 22 January 1999 16:00–18:00

Attempt 6 questions. (If you attempt more, only the best 6 will be counted.) All questions carry the same number of marks. Unless otherwise stated, all groups are finite, and all representations are of finite degree over $\mathbb{C}$.

1. Define a *group representation*. What is meant by saying that 2 representations α, β of G are *equivalent*? Determine all representations of S_3 of degree 2 (up to equivalence), from first principles.

   Show that if α, β are equivalent representations of G then α, β have the same eigenvalues for each $g \in G$. Is the converse true, ie if $\alpha(g), \beta(g)$ have the same eigenvalues for each g, does it follow that α, β are equivalent?

2. What is meant by saying that the representation α is *simple*? Determine all simple representations of D_4 from first principles.

   Show that a simple representation of G necessarily has degree $\leq \|G\|$.

3. What is meant by saying that the representation α is *semisimple*?

   Prove that every representation α of a finite group G (of finite degree over $\mathbb{C}$) is semisimple.

4. Define the *intertwining number* $I(\alpha, \beta)$ of 2 representations α, β.

   Show that if α, β are simple then

$$I(\alpha, \beta) = \begin{cases} 1 & \text{if } \alpha = \beta \\ 0 & \text{if } \alpha \neq \beta. \end{cases}$$

   Hence or otherwise show that the simple parts of a semisimple representation are unique up to order.

5. Define the *character* χ_α of a representation α. Outline the proof that

$$I(\alpha, \beta) = \frac{1}{\|G\|} \sum_{g \in G} \overline{\chi_\alpha(g)} \chi_\beta(g).$$

   Show that two representations α, β of G are equivalent if and only if they have the same character.

6. Show that a simple representation of an abelian group is necessarily of degree 1.

   Prove conversely that if every simple representation of G is of degree 1 then G must be abelian.

7. Draw up the character table of S_4.

   Determine also the *representation ring* of S_4, ie express each product of simple representations of S_4 as a sum of simple representations.

8. Show that a finite group G has only a finite number of simple representations (up to equivalence), say $\sigma_1, \ldots, \sigma_r$; and show that

$$(\deg \sigma_1)^2 + \cdots + (\deg \sigma_r)^2 = \|G\|.$$

   Show that the number of simple representations of S_n of degree d is even if d is odd. Hence or otherwise determine the dimensions of the simple representations of S_5.