Course 424

Group Representations III

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Sam Beckett Theatre Wednesday, 10 June 1991 14:00–16:00

Answer as many questions as you can; all carry the same number of marks.
Unless otherwise stated, all Lie algebras are over $\mathbb{R}$, and all representations are finite-dimensional over $\mathbb{C}$.

1. Define the *exponential* e^X of a square matrix X.

   Determine e^X in each of the following cases:

   $$X = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad X = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} 1 & - \\ 0 & \end{pmatrix}$$

   Which of these 5 matrices X are themselves expressible in the form $X = e^Y$, with (a) Y real, (b) Y complex? (Justify your answers in all cases.)

2. Define a *linear group*, and a *Lie algebra*; and define the Lie algebra $\mathscr{L}G$ of a linear group G, showing that it is indeed a Lie algebra.

   Determine the Lie algebras of $\mathbf{SU}(2)$ and $\mathbf{SO}(3)$, and show that they are isomomorphic.

3. Define a *representation* of a Lie algebra; and show how each representation α of a linear group G gives rise to a representation $\mathscr{L}\alpha$ of $\mathscr{L}G$.

   Determine the Lie algebra of $\mathbf{SL}(2, \mathbb{R})$; and show that this Lie algebra $\mathbf{sl}(2, \mathbb{R})$ has just 1 simple representation of each dimension $1, 2, 3, \ldots$.

4. What is meant by saying that a connected linear group G is *simply-connected*? Show that $\mathbf{SU}(2)$ is simply-connected.

   Sketch the proof that if the linear group G is connected and simply-connected then every representation of $\mathscr{L}G$ lifts to a representation of G.

   Show that if 2 real Lie algebras have the same complexification then their representations (over $\mathbb{C}$) correspond. Hence or otherwise show that all the representations of $\mathbf{sl}(2, \mathbb{R})$ are semisimple.

5. Show that every connected abelian linear group A is isomorphic to

$$\mathbb{T}^m \times \mathbb{R}^n$$

   for some m and n, where $\mathbb{T}$ denotes the torus $\mathbb{R}/\mathbb{Z}$

   Express the multiplicative group $\mathbb{C}^\times$ in this form.