Course 424

Group Representations I

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Exam Hall Friday, 8 March 1991 15:00–17:00

Answer as many questions as you can; all carry the same number of marks.
Unless otherwise stated, all groups are finite, and all representations are finite-dimensional over $\mathbb{C}$.

1. Define a *group representation*. What is meant by saying that 2 representations α, β are *equivalent*? What is meant by saying that the representation α is *simple*?

   Determine all simple representations of S_3 up to equivalence, from first principles.

2. What is meant by saying that the representation α is *semisimple*?

   Prove that every finite-dimensional representation α of a finite group over $\mathbb{C}$ is semisimple.

   Define the *character* χ_α of a representation α.

   Define the *intertwining number* $I(\alpha, \beta)$ of 2 representations α, β. State and prove a formula expressing $I(\alpha, \beta)$ in terms of χ_α, χ_β.

   Show that the simple parts of a semisimple representation are unique up to order.

3. Explain how a representation β of a subgroup $H \subset G$ *induces* a representation β^G of G.

   Show that
   $$\frac{\bar{g}}{|G|}\chi_{\beta^G}(\bar{g}) = \sum_{\bar{h} \subset \bar{g}} \frac{\bar{h}}{|H|}\chi_\beta(\bar{h}).$$

   Determine the characters of S_4 induced by the simple characters of S_3, and so draw up the character table of S_4.

4. Show that the number of simple representations of a finite group G is equal to the number s of conjugacy classes in G.

   Show also that if these representations are $\sigma_1, \ldots, \sigma_s$ then
   $$\dim^2 \sigma_1 + \cdots + \dim^2 \sigma_s = |G|.$$

   Determine the dimensions of the simple representations of S_5, stating clearly any results you assume.

5. Draw up the character table of the alternating group A_4 (the subgroup of S_4 formed by even permutations).

   Determine also the *representation ring* of A_4, ie express each product of simple representations of A_4 as a sum of simple representations.

6. Explain the division of simple representations (over $\mathbb{C}$) into *real*, *essentially complex* and *quaternionic*. Give an example of each (justifying your answers).

   Show that if α is a simple representation with character χ then the value of
   $$\sum_{g \in G} \chi(g^2)$$
   determines which of these 3 types α falls into.