Course 424

Group Representations

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G.M.B. Friday, 22 May 1995 14:00–16:00

Answer as many questions as you can; all carry the same number of marks.
Unless otherwise stated, all representations are finite-dimensional over $\mathbb{C}$.

1. Define a *group representation*. What is meant by saying that 2 representations α, β are *equivalent*? What is meant by saying that the representation α is *simple*?

   Determine all simple representations of D_4 (the symmetry group of a square) up to equivalence, from first principles.

2. What is meant by saying that the representation α is *semisimple*?

   Prove that every finite-dimensional representation α of a finite group over $\mathbb{C}$ is semisimple.

   Define the *character* χ_α of a representation α.

   Define the *intertwining number* $I(\alpha, \beta)$ of 2 representations α, β. State without proof a formula expressing $I(\alpha, \beta)$ in terms of χ_α, χ_β.

   Show that the simple parts of a semisimple representation are unique up to order.

3. Draw up the character table of S_4.

   Determine also the *representation ring* of S_4, ie express each product of simple representations of S_4 as a sum of simple representations.

4. Show that the number of simple representations of a finite group G is equal to the number s of conjugacy classes in G.

Show also that if these representations are $\sigma_1, \dots, \sigma_s$ then

$$\dim^2 \sigma_1 + \cdots + \dim^2 \sigma_s = |G|.$$

Determine the dimensions of the simple representations of S_5, stating clearly any results you assume.

5. Explain the division of simple representations of a finite group G over $\mathbb{C}$ into *real*, *essentially complex* and *quaternionic*. Give an example of each (justifying your answers).

   Show that if α is a simple representation with character χ then the value of

   $$\sum_{g \in G} \chi(g^2)$$

   determines which of these 3 types α falls into.

6. Define a *measure* on a compact space. State carefully, but without proof, Haar's Theorem on the existence of an invariant measure on a compact group. To what extent is such a measure unique?

   Prove that every representation of a compact group is semisimple.

   Which of the following groups are (a) compact, (b) connected:

   $$\mathrm{O}(n), \mathrm{SO}(n), \mathrm{U}(n), \mathrm{SU}(n), \mathrm{GL}(n, \mathbb{R}), \mathrm{SL}(n, \mathbb{R})?$$

   (Justify your answer in each case.)

7. Determine the conjugacy classes in $\mathrm{SU}(2)$.

   Prove that $\mathrm{SU}(2)$ has just one simple representation of each dimension $1, 2, \dots$; and determine the character of this representation.

   If $D(j)$ denotes the simple representation of $\mathrm{SU}(2)$ of dimension $2j+1$, for $j = 0, 1/2, 1, \dots$, express the product $D(j)D(k)$ as a sum of $D(j)$'s.

8. Define the *exponential* e^X of a square matrix X.

   Determine e^X in each of the following cases:

   $$X = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad X = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

   Which of these 5 matrices X are themselves expressible in the form $X = e^Y$, with Y real? (Justify your answers in all cases.)

9. Define a *linear group*, and a *Lie algebra*.

   Define the Lie algebra $\mathscr{L}G$ of a linear group G, and outline the proof that it is indeed a Lie algebra.

   Determine the Lie algebras of SU(2) and SO(3), and show that they are isomomorphic.

10. Define a *representation* of a Lie algebra; and show how each representation α of a linear group G gives rise to a representation $\mathscr{L}\alpha$ of $\mathscr{L}G$.

    Determine the Lie algebra of $\mathrm{SL}(2, \mathbb{R})$; and show that this Lie algebra $\mathrm{slg}(2, \mathbb{R})$ has just 1 simple representation of each dimension $1, 2, 3, \ldots$.