Course 424

Group Representations III

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Arts Block 3051 Thursday, 17 June 1993 14:00–16:00

Answer as many questions as you can[1]*; all carry the same number of marks.*
Unless otherwise stated, all Lie algebras are over $\mathbb{R}$*, and all representations are finite-dimensional over* $\mathbb{C}$.

1. Define the *exponential* e^X of a square matrix X.

   Determine e^X in each of the following cases:

   $$X = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad X = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad X = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}, \quad X = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

   Show that if X has eigenvalues λ, μ then e^X has eigenvalues e^λ, e^μ.

   Which of the above 5 matrices X are themselves expressible in the form $X = e^Y$ for some real matrix Y? (Justify your answers in all cases.)

2. Define a *linear group*, and a *Lie algebra*; and define the Lie algebra $\mathscr{L}G$ of a linear group G, showing that it is indeed a Lie algebra.

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[1] Nb: There is a question overleaf.

Define the *dimension* of a linear group; and determine the dimensions of each of the following groups:

$$\mathbf{O}(n), \mathbf{SO}(n), \mathbf{U}(n), \mathbf{SU}(n), \mathbf{GL}(n, \mathbb{R}), \mathbf{SL}(n, \mathbb{R}), \mathbf{GL}(n, \mathbb{C}), \mathbf{SL}(n, \mathbb{C})?$$

3. Determine the Lie algebras of $\mathbf{SU}(2)$ and $\mathbf{SO}(3)$, and show that they are isomomorphic.

   Show that the 2 groups themselves are *not* isomorphic.

4. Define a *representation* of a Lie algebra $\mathscr{L}$. What is meant by saying that such a representation is (a) *simple*, (b) *semisimple*?

   Determine the Lie algebra of $\mathbf{SL}(2, \mathbb{R})$, and find all the simple representations of this algebra.

   Show that every representation of the group $\mathbf{SL}(2, \mathbb{R})$ is semisimple, stating carefully but without proof any results you need.

5. Show that every connected abelian linear group A is isomorphic to

$$\mathbb{T}^m \times \mathbb{R}^n$$

   for some m and n, where $\mathbb{T}$ denotes the torus $\mathbb{R}/\mathbb{Z}$.

   Show that the groups $\mathbb{T}^m \times \mathbb{R}^n$ and $\mathbb{T}^{m'} \times \mathbb{R}^{n'}$ are isomorphic if and only if $m = m'$ and $n = n'$.