Course 424

Group Representations II

Dr Timothy Murphy

Arts Block A5039 Friday, 8 March 1991 15:00–17:00

Answer as many questions as you can; all carry the same number of marks.
All representations are finite-dimensional over $\mathbb{C}$.

1. Define a *measure* on a compact space. State carefully, but without proof, Haar's Theorem on the existence of an invariant measure on a compact group. To what extent is such a measure unique?

   Which of the following groups are (a) compact, (b) connected:

   $$\mathbf{O}(n), \mathbf{SO}(n), \mathbf{U}(n), \mathbf{SU}(n), \mathbf{GL}(n,\mathbb{R}), \mathbf{SL}(n,\mathbb{R}), \mathbf{GL}(n,\mathbb{C}), \mathbf{SL}(n,\mathbb{C})?$$

   (Justify your answer in each case.)

   Prove that every representation of a compact group is semisimple.

2. Prove that every simple representation of a compact abelian group is 1-dimensional and unitary.

   Determine the simple representations of $\mathbf{SO}(2)$.

   Determine also the simple representations of $\mathbf{O}(2)$.

3. Determine the conjugacy classes in $\mathbf{SU}(n)$.

   Prove that $\mathbf{SU}(2)$ has just one simple representation of each dimension $1, 2, \ldots$; and determine the character of this representation.

   If $D(j)$ denotes the simple representation of $\mathbf{SU}(2)$ of dimension $2j+1$, for $j = 0, 1/2, 1, \ldots$, express the product $D(j)D(k)$ as a sum of $D(j)$'s.

4. Determine the conjugacy classes in $\mathbf{SO}(n)$.

   Prove that $\mathbf{SO}(3)$ has just one simple representation of each odd dimension $1, 3, 5, \ldots$.