Course 424

Group Representations II

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Walton Theatre   Friday, 4 April 2003   11:15–12:45

Attempt 5 questions. (If you attempt more, only the best 5 will be counted.) All questions carry the same number of marks. All representations are finite-dimensional over $\mathbb{C}$, unless otherwise stated.

1. Define the representation $\alpha \times \beta$ of the product-group $G \times H$, where α is a representation of G, and β of H.

   Show that if G and H are finite then $\alpha \times \beta$ is simple if and only if both α and β are simple; and show that every simple representation of $G \times H$ is of this form.

   Show that D_6 (the symmetry group of a regular hexagon) is expressible as a product group
   $$D_6 = C_2 \times S_3.$$

   Let γ denote the 3-dimensional representation of D_6 defined by its action on the 3 diagonals of the hexagon. Express γ in the form
   $$\gamma = \alpha_1 \times \beta_1 + \cdots + \alpha_r \times \beta_r,$$
   where $\alpha_1, \ldots, \alpha_r$ are simple representations of C_2, and $\beta_1, \ldots, \beta_r$ are simple representations of S_3.

2. Prove that the only finite-dimensional division-algebras (or skew-fields) over $\mathbb{R}$ are $\mathbb{R}, \mathbb{C}$ and $\mathbb{H}$.

   Show that the endomorphism-ring $\mathrm{End}(\beta)$ of a simple representation β over $\mathbb{R}$ is a finite-dimensional division-algebra over $\mathbb{R}$; and give examples of three such representations with $\mathrm{End}(\beta) = \mathbb{R}, \mathbb{C}, \mathbb{H}$.

3. Define a *measure* on a compact space. State carefully, and outline the main steps in the proof of, Haar's Theorem on the existence of an invariant measure on a compact group.

   Prove that every representation of a compact group is semisimple.

4. Which of the following groups are (a) compact, (b) connected:

$$\mathrm{O}(n), \mathrm{SO}(n), \mathrm{U}(n), \mathrm{SU}(n), \mathrm{GL}(n, \mathbb{R}),$$
$$\mathrm{SL}(n, \mathbb{R}), \mathrm{GL}(n, \mathbb{C}), \mathrm{SL}(n, \mathbb{C}), \mathrm{Sp}(n), \mathrm{E}(n)?$$

   (Justify your answer in each case. $\mathrm{E}(n)$ is the group of isometries of n-dimensional Euclidean space.)

5. Prove that every simple representation of a compact *abelian* group is 1-dimensional.

   Determine the simple representations of $\mathrm{SO}(2)$.

   Determine the simple representations of $\mathrm{O}(2)$.

6. Determine the conjugacy classes in $\mathrm{SU}(2)$.

   Prove that $\mathrm{SU}(2)$ has just one simple representation of each dimension $1, 2, \ldots$; and determine the character of this representation.

7. Determine the conjugacy classes in $\mathrm{SO}(3)$.

   Prove that $\mathrm{SO}(3)$ has just one simple representation of each odd dimension $1, 3, 5, \ldots$; and determine the character of this representation.